# Denominator Periods, Rational-Value Constraints and Achievement-Set Geometry

*Mersenne subseries and Erdős Problem #257*

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## Abstract

For finite nonempty $F \subseteq \mathbb{N}_{>0}$ at base $b \geq 2$, let $D_F$ be the reduced denominator of $\sum_{n\in F}(b^n-1)^{-1}$. The checked lead theorem is $\operatorname{ord}_{D_F}(b) = \operatorname{lcm}(F)$. If $\operatorname{lcm}(F) \geq 2$, then $\operatorname{lcm}(F) < D_F$; the case $b = 2$, $F = \{1\}$ shows why that hypothesis is needed.

Erdős #257 asks whether $X_A = \sum_{a\in A}(2^a-1)^{-1}$ is irrational for every infinite $A \subseteq \mathbb{N}_{>0}$. It remains open. The coordinate used here is the divisor incidence $\mathrm{sc}_A(n) = \#\{d \mid n : d \in A\}$, for which $X_A = \sum_{n\geq 1} \mathrm{sc}_A(n)2^{-n}$; these coefficients are nonnegative divisor counts and need not be binary digits. If $X_A$ is rational, its integral scaled-tail sequence satisfies an exact long-division recurrence. For infinite $A$ that sequence is unbounded, and every zero window of $\mathrm{sc}_A$ at $c+N$ has length at most $\varepsilon \log_2(N+1)+O_{A,p,\varepsilon,c,v}(1)$. Under the additional hypotheses that $\sum_{a\in A} 1/a$ converges and that the reduced odd denominator part $v > 1$ is coprime to the numerator, rationality also forces

$$\sum_{a\in A} \frac{1}{a} \geq \frac{1}{\operatorname{ord}_v(2)}.$$

At the dyadic endpoint $X_A = p/2^c$, an infinite support instead forces either nonsummability of the reciprocal support terms or reciprocal mass greater than 1. This becomes a contradiction only after an independent proof of summability and mass at most 1. Finally, rational values admit an exact Boolean–Möbius scaled-tail characterisation from which the support is reconstructed. These constraints are not known to be contradictory.

We then record the achievement-set geometry. Every restriction to a set $J$ of allowed exponents preserves unique coding, compactness and nowhere density, is perfect when $J$ is infinite, and has measure $2^{-|F|}$ when the complement $J^c = F$ is finite and measure zero when $J^c$ is infinite. For the squarefree support the incidence is $2^{\omega(n)} - 1$; at every even base two divisibility-first certificate hypotheses fail already at precision $b^2$, while adjoining 1 preserves irrationality and restores arbitrarily long base-2 opening blocks. Since the squarefree values at every power-of-two base are jointly controlled by the literature, that failure is internal to a normalisation rather than to the value itself. Finite support represents neither $1/2$ nor $1/21$: membership of the first is equivalent to unbounded greedy skips, and membership of the second to excluding one explicit fatal aligned branch whose eventual affine-supercapacity regime remains possible. For $1/2$, membership would refute


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#257, while non-membership would eliminate only this one candidate. Tao–Teräväinen's base-2 prime-support theorem and Duverney–Tachiya's power-of-two squarefree theorem are cited prior work, not formalised here. These classifications and scoped frontiers do not solve #257 or either target.

## 1 Introduction and main results

**Problem 1.1** (Erdős #257)**.** For every infinite $A \subseteq \mathbb{N}_{>0}$, is $X_A = \sum_{a\in A} \dfrac{1}{2^a-1}$ irrational?

See Erdős [7, p. 222], Erdős and Graham [8, p. 62], and Erdős [9, p. 105]. Bloom's current catalogue record reproduces the displayed problem and labels it open, while explicitly warning that the status is the website owner's present assessment and may omit relevant literature [10]. We therefore use the catalogue for numbering and current reported status only; the original publications and the later cited papers carry the mathematical claims. Write $w_n = (2^n-1)^{-1}$. Expanding each weight as a geometric series and interchanging the two nonnegative sums gives the coordinate this note works in. With the *divisor incidence* $\mathrm{sc}_A(n) = \#\{d \mid n : d \in A\}$,

$$X_A = \sum_{a\in A} \frac{1}{2^a-1} = \sum_{n\geq 1} \frac{\mathrm{sc}_A(n)}{2^n}. \tag{1}$$

The transform is worth reading carefully, because it is where the arithmetic of the problem enters. The datum is a 0/1 selector, the indicator of $A$; what appears in (1) is not that selector but its divisor transform, a nonnegative integer sequence bounded by the divisor function, $\mathrm{sc}_A(n) \leq d(n)$. So #257 is not a generic question about binary digit sequences. Equation (1) is a power-series representation, not a binary expansion: its coefficients are divisor counts drawn from a single support and may exceed 1. Every theorem below is a statement about sequences of that shape.

A small instance fixes the notation. At $A = \{2,3\}$ the incidence sequence begins

$$\mathrm{sc}_A(1), \dots, \mathrm{sc}_A(12) = 0,1,1,1,0,2,0,1,1,1,0,2,$$

the value 2 occurring at the multiples of 6 and the value 0 at the integers prime to 6, and (1) reads $\frac{1}{3} + \frac{1}{7} = \frac{10}{21}$.

Several statements below hold at every integer base, so we write $X_A(b) = \sum_{a\in A}(b^a-1)^{-1}$ for the value at an integer base $b \geq 2$, with $X_A = X_A(2)$. Base 2 is the case Problem 1.1 asks about and is meant whenever no base is named.

**The finite-support theorem.** For a finite nonempty $F \subseteq \mathbb{N}_{>0}$ and an integer $b \geq 2$, write

$$x_F(b) = \sum_{n\in F} \frac{1}{b^n-1} = \frac{N_F}{D_F} \qquad (\gcd(N_F, D_F) = 1,\ D_F > 0).$$

We use the standard trivial-modulus convention $\mathrm{ord}_1(b) = 1$.

**Theorem 1.2** (finite-period noncollapse)**.** *Let $F \subseteq \mathbb{N}_{>0}$ be finite and nonempty, let $b \geq 2$ be an integer, and let $D_F > 0$ be the denominator of $x_F(b)$ in lowest terms. Then $D_F$ is coprime to $b$, and*

$$\operatorname{ord}_{D_F}(b) = \operatorname{lcm}\{n : n \in F\}.$$

*If moreover* $\operatorname{lcm}(F) \geq 2$*, then* $\operatorname{lcm}(F) < D_F$*.*

The order statement is checked, its reduced-denominator form is checked, coprimality is checked, and the growth clause is checked.

*Status.* The problem treated here is open, and this note does not close it. Every statement below marked as checked is a proposition that the pinned Lean kernel accepts from the sources this note links to, with no `sorry`, no added axiom, and no unchecked evaluation. That is a claim about the formal statement, not about its mathematical interest, its novelty, or the original problem. The unresolved obligations are named exactly, in their own section, and none of the finite computations, reductions, or no-go results here removes one of them.

**Structure.** Section 2 develops the arithmetic of Theorem 1.2. Section 3 collects what rationality would force on an arbitrary infinite support, which is the part of this note that quantifies over every support rather than sampling. Section 4 gives representative supports already known to give irrational values, grouped by the argument that reaches them. Section 5 treats the squarefree support, whose values are known at every power-of-two base and which two of the arguments used here provably cannot reach at any even base. Section 6 gives the unrestricted and support-restricted topology and measure classifications, followed by the exact $1/2$ and $1/21$ frontiers. Section 7 states what remains.

## 2 Finite-support denominator periods

A finite support has a rational value, so for a finite support the question is not irrationality but arithmetic: how large the denominator is, and how the base sits inside it. Theorem 1.2 answers the second exactly and, under its stated hypothesis $\operatorname{lcm}(F) \geq 2$, gives a strict lower bound for the first. The omitted boundary is genuine: at $b = 2$ and $F = \{1\}$ the two quantities are both 1.

Six instances at $b = 2$, which also show what the hypothesis $\operatorname{lcm}(F) \geq 2$ is for:

| $F$ | $x_F(2)$ | $D_F$ | $\operatorname{lcm}(F)$ | $\operatorname{ord}_{D_F}(2)$ |
|---|---|---|---|---|
| $\{1\}$ | $1$ | $1$ | $1$ | $1$ |
| $\{2\}$ | $1/3$ | $3$ | $2$ | $2$ |
| $\{2,3\}$ | $10/21$ | $21$ | $6$ | $6$ |
| $\{1,2,3\}$ | $31/21$ | $21$ | $6$ | $6$ |
| $\{2,6\}$ | $22/63$ | $63$ | $6$ | $6$ |
| $\{4,6\}$ | $26/315$ | $315$ | $12$ | $12$ |

The last two columns agree in every row, which is the order statement. In the worked case $F = \{1,2,3\}$, the rational sum is $1 + 1/3 + 1/7 = 31/21$ and $2^6 \equiv 1 \pmod{21}$,

whereas no smaller positive exponent gives 1. The first row is the only finite nonempty support with $\operatorname{lcm}(F) = 1$, and the only one on which $\operatorname{lcm}(F) < D_F$ fails; that is what the hypothesis $\operatorname{lcm}(F) \geq 2$ excludes.

The content is a noncollapse statement. Clearing denominators over $b^{\operatorname{lcm}(F)}-1$ makes the period at most $\operatorname{lcm}(F)$ immediately; what is not immediate is that cancellation in the numerator cannot bring it below. The mechanism is that each selected exponent $n$ contributes, by the cyclotomic route, a prime-power modulus dividing $b^n - 1$ on which $b$ has order exactly $n$; no single reduction can remove all of them at once. Prime powers rather than primes is not a technicality. At $(b, n) = (2, 6)$ no prime divisor of $2^6 - 1 = 63$ has order 6 — one of the exceptional cases in Zsigmondy's theorem — while 9 does, and it is the prime power that carries the witness. The row $F = \{2, 6\}$ above is that case in full: $D_F = 63 = 9 \cdot 7$, and since 2 has order 2 modulo 3 and order 3 modulo 7, the order 6 can only come from 9. The growth clause then follows formally, since the order divides $\varphi(D_F) < D_F$, where $\varphi$ is Euler's totient.

Two boundaries. This is an unconditional statement about every finite support, not a bounded table of examples, and not an implication with an open hypothesis; but it settles no infinite support, and no limit of it does. Denominator-period control is part of the classical method behind Erdős's 1948 argument [6]. Whether this exact sharp form appears in the literature has not been assessed, and no novelty is claimed.

## 3 Rational values and scaled tails

Section 4 below lists supports for which irrationality is known. This section is the other half, and the half that meets Problem 1.1 rather than sampling it: statements that hold for *every* infinite support whose value is rational. The base is 2 throughout, and each statement carries its own hypotheses.

**Integral scaled tails, and unboundedness.** Binary long division turns a hypothetical rational value into an integer recurrence. Fix an integer $v \geq 1$ and a sequence $f\colon \mathbb{N}_{>0} \to \mathbb{N}$ of coefficients. Call an integer sequence $u\colon \mathbb{N} \to \mathbb{Z}$ a *tempered scaled-tail sequence* for $f$ with multiplier $v$ if

$$u(n+1) = 2u(n) - vf(n+1) \quad \text{for every } n \geq 0, \qquad \text{and} \qquad u(n) = o(2^n).$$

The recurrence is one step of long division in base 2: double the remainder, then pay out the next coefficient. The growth condition is what pins the solution down, since two solutions of the recurrence differ by $C\,2^n$ for a constant $C$, and only $C = 0$ survives $u(n) = o(2^n)$. For any coefficient sequence with $f(n) \leq n$ — and $\mathrm{sc}_A(n) \leq d(n) \leq n$ qualifies, $d$ being the divisor function — the series $\sum f(n)2^{-n}$ is rational exactly when a tempered scaled-tail sequence exists for some $v \geq 1$, and every such sequence is then the scaled tail $u(n) = v\sum_{j\geq 1} f(n+j)2^{-j}$, so the orbit is unique rather than merely available (checked). The next theorem exhibits such an orbit for $X_A$ itself, with the coefficient sequence shifted past the power of 2 in the denominator.

**Theorem 3.1** (unbounded scaled-tail states)**.** *Let $A \subseteq \mathbb{N}_{>0}$ be infinite with $X_A = p/(2^c v)$, $v \geq 1$. Then there is $u\colon \mathbb{N} \to \mathbb{N}_{>0}$ with*

$$u(n) = v \sum_{j \geq 1} \mathrm{sc}_A(c + n + j) 2^{-j}$$

*satisfying the exact recurrence*

$$u(n+1) + v\ \mathrm{sc}_A(c + n + 1) = 2u(n), \qquad u(n) \equiv p\, 2^n \pmod{v},$$

*and $u$ is unbounded.*

Checked as Lean. The useful clause is exact: these scaled-tail states cannot remain bounded. In particular they cannot eventually cycle through a finite set of states.

**Divisor coverage cannot have long gaps.** Say that $\mathrm{sc}_A$ has a *zero window* of length $h$ at $N$ if $\mathrm{sc}_A(N+1) = \cdots = \mathrm{sc}_A(N+h) = 0$, that is, if no element of $A$ divides any of $h$ consecutive integers. At $A = \{2, 3\}$, for instance, $\mathrm{sc}_A$ vanishes exactly on the integers prime to 6, so every zero window has length at most 1: one of any two consecutive integers is even.

**Theorem 3.2** (sublogarithmic zero windows)**.** *Let $A \subseteq \mathbb{N}_{>0}$ be nonempty with $X_A = p/(2^c v)$, $v \geq 1$. For every $\varepsilon > 0$ there is a constant $B$, depending on $A, p, \varepsilon, c$ and $v$, such that every zero window of $\mathrm{sc}_A$ at $c + N$ has length*

$$h \leq \varepsilon \log_2(N+1) + B.$$

*For fixed $A, p, c, v$ and $\varepsilon$, the same $B$ works uniformly in $N$ and $h$; no common constant over supports or numerators is asserted.*

Checked as Lean. Read as a constraint on counterexamples, this rules out zero windows of any fixed positive proportion of $\log_2 N$ once $N$ is large. The proof compares an exponential lower bound forced on the scaled tail with an upper bound for divisor sums that grows more slowly than any fixed power of $N$.

**A reciprocal-sum lower bound.** Let $\mathrm{ord}_v(2)$ denote the multiplicative order of 2 modulo an odd $v > 1$, and write the reciprocal mass of $A$ for $\sum_{a \in A} 1/a$.

**Theorem 3.3** (reciprocal mass bound)**.** *Let $A \subseteq \mathbb{N}_{>0}$ be such that $\sum_{a \in A} 1/a$ converges, and suppose $X_A = p/(2^c v)$ with $v > 1$ odd and $\gcd(p, v) = 1$. Then*

$$\sum_{a \in A} \frac{1}{a} \geq \frac{1}{\mathrm{ord}_v(2)}.$$

Checked as Lean. A finite support already supplies an instance: $A = \{2, 3\}$ has $X_A = 10/21$, so $c = 0$, $v = 21$, $\gcd(p, v) = 1$ and $\mathrm{ord}_{21}(2) = 6$, and the bound reads $\frac{1}{2} + \frac{1}{3} \geq \frac{1}{6}$. The summability hypothesis is doing real work: a rational value with fixed odd denominator part $v$ and a convergent reciprocal sum requires mass at least $1/\mathrm{ord}_v(2)$. The statement does not apply when the reciprocal sum diverges, and it gives no positive lower bound uniform in $v$; it constrains the pair (support, denominator) jointly.

The critical dyadic case has a different endpoint.

**Theorem 3.4** (dyadic reciprocal-mass endpoint)**.** *Let $A \subseteq \mathbb{N}_{>0}$ be infinite and suppose $X_A = p/2^c$ for some $p \in \mathbb{Z}$ and $c \in \mathbb{N}$. Then the reciprocal support terms are not summable, or*

$$\sum_{a \in A} \frac{1}{a} > 1.$$

Checked as Lean. The two alternatives are an exact necessary consequence of a dyadic rational value. They do not contradict rationality by themselves: a contradiction requires separate proofs that the reciprocal terms are summable and that the mass is at most 1. The formal nonsummability alternative also does not, by itself, assert a particular asymptotic law for the partial sums.

**A Boolean–Möbius characterisation.** The preceding theorems give necessary conditions. The next statement is an equivalence, and is the sharpest description of rational-valued supports the development has. It removes the support from the description altogether, which is possible because the support can always be read back off the incidence sequence: with $\mu$ the Möbius function, $*$ Dirichlet convolution $(g * h)(n) = \sum_{d \mid n} g(d)h(n/d)$ and $\mathbf{1}_A$ the indicator of $A$, the identity $\mathrm{sc}_A = \mathbf{1}_A * 1$ inverts to $\mu * \mathrm{sc}_A = \mathbf{1}_A$.

For $p \in \mathbb{Z}$, $q \geq 1$ and $U \colon \mathbb{N} \to \mathbb{Z}$, set

$$Q_U(0) = 0, \qquad Q_U(n) = \frac{2U(n-1) - U(n)}{q} \quad (n \geq 1).$$

Call $U$ an *admissible Boolean–Möbius scaled-tail sequence* for $(p, q)$ if

$$\begin{gathered} U(0) = p, \quad U(N) > 0, \quad U(N) \leq q(2\sqrt{N} + 4) \quad (N \geq 0), \\ q \mid 2U(N) - U(N+1) \quad (N \geq 0), \qquad (\mu * Q_U)(n) \in \{0, 1\} \quad (n \geq 1). \end{gathered} \tag{2}$$

The divisibility condition makes $Q_U$ integral; the last condition says that Möbius inversion of the quotient is the indicator of a set.

**Theorem 3.5** (Boolean–Möbius scaled-tail correspondence)**.** *Let $p \in \mathbb{Z}$ and $q \geq 1$. There exists a support $A$ with $0 \notin A$, with some positive element, and with $X_A = p/q$, if and only if there exists an admissible Boolean–Möbius scaled-tail sequence $U$ for $(p, q)$. In that case the support is recovered as*

$$A = \{n : (\mu * Q_U)(n) = 1\},$$

*with $Q_U$ as in (2).*

Checked as Lean. The finite example $A = \{2, 3\}$ makes the definition concrete. Here $p/q = 10/21$, and

$$Q_U(1), \dots, Q_U(6) = 0, 1, 1, 1, 0, 2, \qquad U(0), \dots, U(6) = 10, 20, 19, 17, 13, 26, 10.$$

Möbius inversion gives $(\mu * Q_U)(1), \dots, (\mu * Q_U)(6) = 0, 1, 1, 0, 0, 0$, the indicator of $\{2, 3\}$ through that range. This finite calculation illustrates the correspondence; the admissible scaled-tail sequence itself is an infinite object, and the finite support is not a counterexample to Problem 1.1.

One boundary on the equivalence. The support it produces is not required to be infinite, and a finite support supplies an admissible scaled-tail sequence for its own value, so existence of such a sequence is not by itself a counterexample to Problem 1.1. What the equivalence changes is the search space, not the difficulty.

**Combined constraints on a rational counterexample.** Taken together: a counterexample to Problem 1.1 would have an unbounded scaled-tail sequence obeying an exact linear recurrence and sublogarithmic divisor-coverage gaps. If its reciprocal sum converged and its reduced odd denominator part were greater than 1, it would also satisfy Theorem 3.3; at a dyadic rational value it would instead satisfy Theorem 3.4. Theorem 3.5 reconstructs every rational value, but by itself does not force the reconstructed support to be infinite. These qualifications matter: the statements have different hypotheses, and no jointly contradictory combination has been proved.

## 4 Representative known irrational supports

The following table is representative, not exhaustive. It groups the displayed supports by five mechanisms rather than suggesting that its rows classify all known cases.
In the sentence immediately following the p. 222 theorem, Erdős says that pairwise coprimality can be removed by a more complicated argument, but he does not give that argument; p. 226 repeats that boundary. Accordingly the table uses only the fully printed pairwise-coprime theorem, not the stronger unproved-in-print extension.
Five remarks on the table: one on containment between the mechanisms, three on the literature rows, and one on what "checked here" means.

*The mechanisms are not nested, and one non-containment is proved.* The base-$b$ full-support theorem is exactly the $A = \mathbb{N}_{>0}$ statement; the multiples rows genuinely specialise it after a base change, but the sparse rows do not, and this is not merely an artefact of how they were proved. The denominator-gap criterion behind the factorial and power-of-two rows *provably cannot* reach the full support: the prefix lcm grows too slowly for its hypothesis to hold (checked). Conversely the analytic method of [3] reaches the primes, which are neither eventually periodic nor pairwise coprime with summable reciprocals. No list contains another, and their union does not exhaust the infinite supports.

*One theorem of Duverney and Tachiya generates several rows.* Their refinement of the Chowla–Erdős method [2, Cor. 1.2, author-preprint p. 4; proof pp. 9–11] proves, for a pairwise coprime sequence $E$ of polynomial growth and the set $F_s(E)$ of products of its members with exponents below $s$, that 1 and the values $\sum_{n\in F_s}(q^{jn^i}-1)^{-1}$ are linearly independent over $\mathbb{Q}$ whenever $|q|^L \le s$, with $L = \operatorname{lcm}(1,\dots,\ell)$ and $\ell$ bounding the exponent $i$. This is a row-generating theorem, not an isolated example: with $E$ the primes, $s=\infty$ returns the full support at every base, while $s=2$ returns the squarefree support. For $s=2$, the constraint forces $\ell = 1$ and $|q| \le 2$, but the free exponent $j$ gives every base $b = 2^j$. Thus this route excludes bases that are not powers of 2, rather than all bases $b \ge 3$; it also excludes higher monomial degrees $i \ge 2$. Their conclusion is stronger than irrationality, being linear independence of the whole finite family across the chosen exponents $j$.

| Support $A$ | Bases | Mechanism and authority |
|---|---|---|
| *Classical full support, and base change into it* | | |
| all of $\mathbb{N}_{>0}$ | every $b \geq 2$ | Erdős 1948 [6]; formalised here |
| multiples of a fixed $d$ | every $b \geq 2$ | dilation: the multiples series at base $b$ *is* the full-support series at base $b^d$ (checked) |
| *Periodicity of the coefficient sequence* | | |
| eventually periodic | every $b \geq 2$ | checked here; Luca–Tachiya prove the nonnegative purely-periodic case [4, Theorem 1, p. 139; proof pp. 149–150]; finite rational prefixes give the infinite eventual case |
| a residue class; the odd numbers | every $b \geq 2$ | special cases of the row above (residue class, odd); Luca–Tachiya's Example 2 strengthens the odd row to joint linear independence of every finite divisor-convolution ladder, also for negative integer bases with absolute value greater than one [4, Example 2, p. 140] |
| *Denominator gaps along a sparse support* | | |
| factorials $\{n!\}$ | every $b \geq 2$ | checked here |
| powers of two $\{2^k\}$ | every $b \geq 2$ | checked here |
| pairwise coprime, $\sum_{a \in A} a^{-1} < \infty$ | every $b \geq 2$ | Erdős [7], theorem on p. 222; formalised here |
| *Multiplicative closure: Chowla–Erdős, refined* | | |
| $F_s(E)$ and the monomial images $\{jn^i : n \in F_s(E)\}$ | $b = q^j$ with $\lvert q\rvert^L \leq s$, $L = \operatorname{lcm}(1, \ldots, \ell)$ | Duverney–Tachiya [2, Cor. 1.2, author-preprint p. 4]; linear independence, not only irrationality; not formalised here |
| $s$-free positive integers, with or without 1 | $b = q^j$, $2 \leq \lvert q\rvert \leq s$, $j \geq 1$ | *ibid.*, with $E$ the primes and $\ell = i = 1$; deleting 1 changes the value by $1/(b-1) \in \mathbb{Q}$ |
| squarefree | every $b = 2^j$, $j \geq 1$ | Duverney–Tachiya [2, Cor. 1.2 and Ex. 1.1, author-preprint p. 4]; joint linear independence for every finite set of such bases; not formalised here |
| coprime to a fixed $N$; sums of two squares | every $b \geq 2$ | *ibid.*, Examples 1.3 and 1.2; not formalised here |
| *Analytic* | | |
| primes | $b = 2$ proved | Tao–Teräväinen [3], Thm. 1.3, p. 4; proof pp. 44–56; not formalised here |
| primes, $b \geq 3$; prime powers | — | *ibid.*, asserted as modifications with details omitted in the cited version |
| arbitrary infinite $A$ | — | **Open** (Problem 1.1) |

*The classical full-support value now has a digit-level refinement.* At base 2, Campbell writes the Erdős–Borwein constant in the equivalent forms

$$E = \sum_{n\geq 1} \frac{1}{2^n - 1} = \sum_{n\geq 1} \frac{d(n)}{2^n}$$

and proves that the binary block 11 occurs infinitely often in its base-2 expansion [1, Theorem 1, p. 12; proof pp. 12–24]. This adds genuine digit-distribution information to the full-support row, but it neither proves normality nor addresses an arbitrary infinite support $A$, so it does not change the open status of Problem 1.1.

*The two analytic rows do not have the same standing.* Theorem 1.3 on p. 4, proved in Section 5 on pp. 44–56, of [3] proves the prime-support case at base 2, the series there being $\sum_{n\geq 1} \omega(n)/2^n$. The extension to every integer base, and the prime-power support — which those authors themselves identify with Problem 1.1 — are asserted by remark, with the modifications explicitly left to the reader. The table keeps the three apart, and only the first is proved.

*Formalisation is not priority.* Rows marked "checked here" are Lean statements accepted by the pinned kernel. For the full support that is a formalisation of Erdős; Luca and Tachiya's RIMS paper already proves the nonnegative purely-periodic case; a finite rational-prefix correction gives the eventual-periodic extension used here [4, Theorem 1, p. 139; proof pp. 149–150], while Theorem A restates the broader signed purely-periodic theorem without reproducing its earlier proof. No priority is claimed anywhere in this table.

*Remark* (signed periodic weights: a gap in the method, not in the mathematics). Dropping nonnegativity weakens the conclusion available, and it is worth being exact about where. For $u\colon \mathbb{N}_{>0} \to \mathbb{Z}$ periodic and $b \geq 2$, what is checked here is a *dichotomy*: the series $\sum_{n\geq 1} u(n)/(b^n - 1)$ is irrational, *or* some power of $b$ times its value is an integer (checked). For a nonnegative support indicator the terminating alternative is excluded by the rows above; for mixed signs this argument does not exclude it.

The terminating alternative never in fact occurs. Theorem A in Luca and Tachiya's 2017 RIMS paper explicitly restates their earlier Theorem 1.1 in the exact form needed here: $\sum_{n\geq 1} a_n/(q^n - 1)$ is irrational for every integer $q$ with $|q| > 1$ and every nonzero purely periodic integer sequence $(a_n)$ [4, Theorem A, p. 139]. Thus the disjunction above is an artefact of the route taken, not a feature of the problem: the second branch is empty, and their author restatement is what shows it. The RIMS paper does not reproduce the earlier proof or verify the broader eventual-rational formulation, so neither is claimed here. The local statement retains only the interest of being an independently checked argument with a visible boundary. Section 5 records a second, sharper instance of the same phenomenon.

## 5 Squarefree support and a coordinate-dependent obstruction

Let $A_{\mathrm{sf}} = \{d \geq 2 : d \text{ squarefree}\}$: a support of density $6/\pi^2$, far denser than the primes, and not periodic. Its values at all power-of-two bases are jointly linearly independent with 1, by a theorem of Duverney and Tachiya recalled below. This section establishes

that two block-certificate arguments used here cannot reach that support at any even base, and that the reason lies in a normalisation rather than in the value.

**Theorem 5.1** (squarefree divisor incidence)**.** *For every $n \geq 1$ the number of squarefree divisors of $n$ is $2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors, and hence*

$$\mathrm{sc}_{A_{\mathrm{sf}}}(n) = 2^{\omega(n)} - 1.$$

*In particular* $\mathrm{sc}_{A_{\mathrm{sf}}}(n)$ *is* odd *for every $n \geq 2$.*

The count is checked, the incidence formula is checked, and the parity conclusion is checked. The proof is the bijection between squarefree divisors of $n$ and subsets of its prime factors (Lean), so the count is $2^{\omega(n)}$; removing $d = 1$ leaves an odd number whenever $\omega(n) \geq 1$. At $n = 12 = 2^2 \cdot 3$ the squarefree divisors are $1, 2, 3, 6$, so $\mathrm{sc}_{A_{\mathrm{sf}}}(12) = 3$; at $n = 30$ they are the eight divisors of 30 and $\mathrm{sc}_{A_{\mathrm{sf}}}(30) = 7$.

### 5.1 The values are known at every power-of-two base

Duverney and Tachiya's Corollary 1.2, applied with $E$ the primes, $s = 2$ and $\ell = 1$, gives $L = \mathrm{lcm}(1) = 1$ and the admissibility condition $|q| \leq 2$. The exponent $j$ in their displayed family remains arbitrary. Thus, for every $h \geq 1$, their Example 1.1 says that the numbers

$$1, \qquad \sum_{n \geq 1} \frac{|\mu(n)|}{2^{jn} - 1} \quad (1 \leq j \leq h)$$

are linearly independent over $\mathbb{Q}$ [2, Cor. 1.2 and Ex. 1.1, author-preprint p. 4]. Since $|\mu|$ is the indicator of the squarefree integers including $n = 1$, and $A_{\mathrm{sf}}$ omits only $n = 1$, whose weight at base $2^j$ is $(2^j - 1)^{-1} \in \mathbb{Q}$,

$$X_{A_{\mathrm{sf}}}(2^j) = \sum_{n \geq 1} \frac{|\mu(n)|}{2^{jn} - 1} - \frac{1}{2^j - 1},$$

and rational translation preserves the joint linear independence with 1.

**Corollary 5.2** (joint power-of-two-base theorem)**.** *For every $h \geq 1$, the $h + 1$ numbers*

$$1, \quad X_{A_{\mathrm{sf}}}(2), \quad X_{A_{\mathrm{sf}}}(4), \quad \ldots, \quad X_{A_{\mathrm{sf}}}(2^h)$$

*are linearly independent over $\mathbb{Q}$. In particular, every $X_{A_{\mathrm{sf}}}(2^j)$, $j \geq 1$, is irrational.*

Two boundaries on that. It is a citation, not a formalisation: nothing in this development proves it. The admissibility condition $|q|^L \leq s$ at $s = 2$ allows only $|q| = 2$, so this citation covers the bases $2^j$ and does not cover bases that are not powers of 2; this note proves no result about those remaining values.

### 5.2 Two block-certificate hypotheses have no instance

Fix $b \geq 2$ and a nonnegative coefficient sequence $f$. The two hypotheses used below are best printed rather than named. For every precision $q \geq 1$, they ask for $N, K, L, C \in \mathbb{N}$ with $K \leq L$ satisfying the common conditions

$$\sum_{r=K+1}^{L} f(N+r)b^{L-r} \leq C, \qquad \exists t \in \mathbb{N} : f(N+L+1+t) > 0, \qquad q(C+N+L+2) < b^L, \tag{3}$$

together with one of the two first-block conditions

$$\begin{aligned} &\text{digitwise:} \quad b^r \mid f(N+r) \quad (1 \leq r \leq K), \\ &\text{weighted:} \quad b^K \mid \sum_{r=1}^{K} f(N+r)b^{K-r}. \end{aligned} \tag{4}$$

Thus the data $N, K, L, C$ may depend on $q$. The parity of Theorem 5.1 refutes both alternatives for the squarefree incidence sequence at every even base.

**Corollary 5.3** (neither divisibility-first hypothesis has an instance)**.** *For the squarefree support $A_{\mathrm{sf}}$ and every even base $b \geq 2$, neither the digitwise nor the carry-aware block-certificate hypothesis holds. Both fail already at precision $q = b^2$.*

*Proof.* The hypothesis quantifies over every precision, so it is enough to exhibit one precision at which no admissible data exist; we take $q = b^2$ and split on whether the first block is empty.

Both hypotheses ask, for every precision $q$, for $N$, $K \leq L$ and $C$ with a first-block condition on $\mathrm{sc}_{A_{\mathrm{sf}}}(N+1), \ldots, \mathrm{sc}_{A_{\mathrm{sf}}}(N+K)$, a middle bound $\sum_{r=K+1}^{L} \mathrm{sc}_{A_{\mathrm{sf}}}(N+r)b^{L-r} \leq C$, a nonzero coefficient beyond $N+L$, and $q(C+N+L+2) < b^L$. Take $q = b^2$ and write $f = \mathrm{sc}_{A_{\mathrm{sf}}}$.

Suppose $K \geq 1$. The digitwise condition requires $b^r \mid f(N+r)$ for $1 \leq r \leq K$. If $N + K \geq 2$, its instance at $r = K$ makes the even number $b$ divide the odd number $f(N+K)$. The carry-aware condition requires $b^K \mid \sum_{r=1}^{K} f(N+r)b^{K-r}$; reducing modulo $b$ kills every term but $r = K$, so the same contradiction follows when $N + K \geq 2$.

The only remaining case with $K \geq 1$ is $N = 0, K = 1$. If $L = 1$, then $q(C+N+L+2) \geq 3b^2 > b$; if $L \geq 2$, the $r = 2$ term in the middle sum is $f(2)b^{L-2} = b^{L-2}$, so $qC \geq b^L$. Both contradict the strict size inequality.

Suppose $K = 0$, where both first-block conditions are vacuous. If $L = 0$ the middle sum is empty and $q(C+N+2) < b^0 = 1$ is impossible. If $L \geq 1$ the $r = 1$ term gives $C \geq b^{L-1}$ whenever $N \geq 1$, and again the size inequality fails. If $N = 0$ and $L = 1$, its left side is at least $3b^2 > b$; if $N = 0$ and $L \geq 2$, the $r = 2$ term gives $C \geq b^{L-2}$ and hence $qC \geq b^L$. These exhaust the cases. □

The two engine-facing nonexistence statements are checked directly as the carry-aware no-go and the digitwise no-go. The displayed proof is retained to expose the empty- and one-position-block boundary cases rather than leaving the scope hidden behind the interfaces.

### 5.3 The obstruction is a normalisation, not the value

Corollary 5.3 is a statement about two arguments failing on a value that Corollary 5.2 shows to be irrational. The question it raises is therefore not whether the value can be reached, but what the failure is a property *of*. It is a property of where the support starts.

Adjoin 1 to the support and write $A_{\mathrm{sf}}^{+} = A_{\mathrm{sf}} \cup \{1\}$, the full squarefree support. Then

$$X_{A_{\mathrm{sf}}^{+}}(b) - X_{A_{\mathrm{sf}}}(b) = \frac{1}{b-1} \in \mathbb{Q},$$

so the two supports pose the same irrationality question at every base, while the divisor incidence changes from $2^{\omega(n)} - 1$ to $2^{\omega(n)}$ — from odd to even at every $n \geq 2$. The parity obstruction of Corollary 5.3 evaporates under a shift that provably cannot change the answer. The shifted coefficient identity and the exact equivalence of the two irrationality questions are checked as $2^{\omega(n)}$ incidence and the shift iff. More is true: the shifted first-block condition at base 2 asks for $2^r \mid 2^{\omega(N+r)}$, that is $\omega(N+r) \geq r$ for $1 \leq r \leq K$, and a Chinese-remainder construction reserving $r$ fresh primes for each shift $r$ supplies such an $N$ for every $K$. This is checked both as the arithmetic block theorem for $\omega$ and in the engine-facing form for the shifted incidence.

It does *not* follow that either argument certifies the shifted support: the opening block is one of the conditions listed above, and the middle bound and the arithmetic inequality are untouched by this observation. What does follow is a methodological point: an obstruction stated against a coefficient sequence can be an artefact of the normalisation chosen for that sequence. Before a no-go result is reported as a property of a problem, the coordinate it is stated in should be varied by a transformation the problem is known to be invariant under. Here the transformation is adding one rational number, and it removes the obstruction entirely.

The same invariance holds for every finite change, not only this one. If $A \triangle B$ is finite, choose $M$ above all of its elements. The two support series then have the same tail beyond $M$, while each omitted prefix is rational. The two directions of this argument are exactly the checked prefix lemmas Lean and Lean. Thus $X_A(b)$ is irrational if and only if $X_B(b)$ is irrational for every integer $b \geq 2$. The present shift is the smallest instance of a general checked finite-change principle.

This is the second time in this note that a boundary turns out to belong to the method rather than to the mathematics. In Remark 4 the terminating alternative of the signed periodic dichotomy is empty, and a theorem of Luca and Tachiya is what shows it; here a parity obstruction survives only until the support is shifted by one element. In each case the correction came from outside the development — once from the literature, once from asking what the statement was invariant under. A formalised no-go result carries exactly the authority of its hypotheses, and its hypotheses include the coordinates it was written in.

## 6 Achievement-set geometry and the value $1/2$

Problem 1.1 asks whether every value obtainable from infinitely many of the weights $w_n = (2^n - 1)^{-1}$ is irrational. The set of all values obtainable, from finite and infinite

selections alike, is the achievement set $\mathcal{A} = \{\sum_{n\geq 1} \varepsilon_n w_n : \varepsilon_n \in \{0,1\}\}$, and this section is about its geometry.

**Theorem 6.1** (geometry of $\mathcal{A}$)**.** *$\mathcal{A}$ is compact, perfect, totally disconnected and nowhere dense, and has Lebesgue measure* 1.

Checked as compact, perfect, totally disconnected, nowhere dense, and of measure one. So $\mathcal{A}$ is a fat Cantor set: strict tail domination $\sum_{\ell>n} w_\ell < w_n$ opens a gap at every level, while the total measure is not lost. The division of credit is exact. The strict inequality, the resulting distinctness of subsums over distinct supports, and the Cantor conclusion for every integer base are Remark 4.1 (p. 13) of Kovač and Tao [5]; no novelty is claimed for any of the three. That remark makes no metric assertion, and strict tail domination does not determine the measure: the weights $3^{-n}$ satisfy $\sum_{\ell>n} 3^{-\ell} = 3^{-n}/2 < 3^{-n}$ and produce a null achievement set, the base-3 digits-in-$\{0,1\}$ Cantor set. So the measure-one clause is added here rather than formalised from there, and it is the arithmetic of the Mersenne weights that supplies it. With $T_n = \sum_{k>n} w_k$, the standard level-$n$ convex-hull cover consists of $2^n$ disjoint intervals of length $T_n$, its nested intersection is $\mathcal{A}$, and

$$2^n T_n = \sum_{j\geq 1} \frac{2^n}{2^{n+j}-1} \longrightarrow \sum_{j\geq 1} 2^{-j} = 1,$$

dominated by $2^{1-j}$. Continuity of measure from above therefore gives $\lambda(\mathcal{A}) = 1$.

Membership is characterised level by level, by a greedy expansion. Run through $n = 1, 2, \ldots$ carrying a remainder, initially the target $x$, and at level $n$ subtract $w_n$ from the remainder if $w_n$ does not exceed it, leaving the remainder unchanged otherwise; call a level at which nothing is subtracted *skipped*. Say that the expansion *survives* level $n$ if the remainder after that level is at most the remaining mass $T_n$. Then $x \in \mathcal{A}$ if and only if $x \geq 0$ and the expansion survives every level (Lean). Non-membership of a nonnegative target is therefore visible at a single level, and for a rational target strict failure can be certified effectively: rational upper bounds for the remaining tail converge to $T_n$, so once one lies below the rational residual the fatal inequality is proved. The finite example below uses an exact rational upper bound. For $x = 3/4$, the greedy algorithm skips $w_1 = 1$ and leaves residual $3/4$, while the exact zero-lookahead upper bound for the remaining tail is $2w_2 = 2/3 < 3/4$; hence $3/4 \notin \mathcal{A}$ (Lean).

The selector-to-subsum map sending a 0/1 string $(\varepsilon_n)$ to $\sum_n \varepsilon_n w_n$, which the strict tail inequality makes injective, remains injective after restriction to any subfamily. That matters because Problem 1.1 quantifies over every infinite support rather than over $\mathbb{N}_{>0}$. For a set $J$ of future offsets write $T_J(n) = \sum_{k\in J} w_{n+k+1}$ for the tail restricted to $J$ (definition, summable at Lean). Then $T_J(n) < w_n$ for every $J$ and every $n \geq 1$ (checked), since deleting weights only shrinks a tail that already sits below $w_n$ at the full support. Consequently the digit map is injective on strings supported in $J$ (checked), stated on the subtype of digit strings vanishing off $J$ and evaluated by the restricted digit map, so the statement is about the subseries itself and not a projection of the full one. Thus uniqueness of the selector survives passage to an arbitrary subfamily; this is an injectivity statement, not a statement about binary digits of the value. That is a constraint on the shape of an argument, not on the supports: it decides no value, and it is weaker than any statement in Section 3.

### 6.1 Geometry after restricting the allowed exponents

The injectivity statement has a geometric completion for every set $J \subseteq \mathbb{N}$ of allowed digit positions. Let

$$\mathcal{A}_J = \left\{ \sum_{k \in J} \varepsilon_k w_{k+1} : \varepsilon_k \in \{0,1\} \right\}.$$

Formally, the allowed strings form the supported digit set, which is closed. Its range is the restricted achievement set; the range and image descriptions agree by the image theorem. Consequently $\mathcal{A}_J$ is compact and closed. It lies inside $\mathcal{A}$ by support restriction and is therefore nowhere dense. If $J$ is infinite, the digit space is preperfect; injectivity transfers that property to $\mathcal{A}_J$, so the closed set is perfect. Thus every infinite allowed support gives a compact perfect nowhere-dense set with a unique selector for each point.

The metric classification is exact. The summability lemma controls digit terms, and changing one coordinate changes the value by precisely its signed weight by the update formula. If $k \notin J$, allowing $k$ splits the new set into $\mathcal{A}_J$ and its translate by $w_{k+1}$ exactly. The two pieces are disjoint, so adjoining one coordinate doubles the volume. At $J = \mathbb{N}$, the restricted set is the full set $\mathcal{A}$ by definition.

**Theorem 6.2** (volume dichotomy for every allowed support)**.** *For every $J \subseteq \mathbb{N}$, exactly one of the following measure formulas applies:*

$$J = F^c \text{ for a finite } F, \quad \lambda(\mathcal{A}_J) = 2^{-|F|},$$
$$J^c \text{ is infinite}, \qquad \lambda(\mathcal{A}_J) = 0.$$

For finite $F$, the division-free identity multiplies the face volume by $2^{|F|}$ and its solved form gives $2^{-|F|}$. Monotonicity under enlarging $J$ is checked. It traps a set with infinitely many forbidden coordinates below finite-codimension faces of arbitrarily small dyadic measure, yielding measure zero. The two cases are assembled in the formal dichotomy. This theorem classifies the size of the value set generated inside any prescribed family of exponents. It does not classify the arithmetic nature of its individual points and therefore does not settle Problem 1.1.

### 6.2 The value $1/2$

A rational point of $\mathcal{A}$ attained by an infinite support refutes Problem 1.1. The distinguished candidate is $1/2$, and Theorem 6.3 gives a self-contained pair of exact alternatives.

The greedy expansion of $1/2$ is an exact finite computation as far as one cares to take it. Through level 21 it takes the exponents

$$2,\ 3,\ 6,\ 7,\ 14,\ 20,\ 21$$

and skips the others, the skipped runs being $\{1\}$, $\{4,5\}$, $\{8,\dots,13\}$ and $\{15,\dots,19\}$; every level through 21 is survived. The second condition below asks whether the list of skipped exponents is infinite, and no finite computation answers that.

In the statement, $u = (u_1,\dots,u_d) \in \{0,1\}^d$ is a finite prefix, $V(u) = \sum_{n \le d} u_n w_n$ is its value, and $T_n = \sum_{k>n} w_k$ as above.

**Theorem 6.3** (membership and non-membership at $1/2$)**.** *The value $1/2$ belongs to $\mathcal{A}$ if and only if its canonical greedy expansion omits infinitely many exponents. Dually, $1/2 \notin \mathcal{A}$ is equivalent to the existence of a finite* fatal gap *— a prefix $u$ through rank $d$ with $V(u) + T_{d+1} < \frac{1}{2} < V(u) + w_{d+1}$, which makes every continuation miss — and to the greedy orbit having a last skipped exponent. Moreover no finite support has value $1/2$.*

Checked as the greedy form, the terminal-bit form, the skipped-rank form, the fatal-gap equivalence, its transfer to non-membership, and the finite-support exclusion.

The formal source also proves equivalent terminal-bit and unbounded skipped-rank formulations in its internal half-cylinder seam coordinate; the two middle source links above are those versions. Their definitions are not needed for the self-contained greedy and fatal-gap statement used here. The two sides are asymmetric: non-membership is witnessed by a *finite* fatal gap and is therefore semi-decidable, whereas membership is an infinite condition. Computing further can only raise a lower bound on where a fatal gap could occur; it cannot establish survival.

One natural route to $1/2$ is closed. The Boolean support selected by the negative values of the Möbius function has value exactly $1/2$ plus the positive Möbius tail, hence at least $1/2 + 1/63$ (identity, bound). That closes the sign-truncation construction; it excludes no other support.

### 6.3 A second rational target

The target $1/21$ has a useful property that does not depend on any finite search.

**Theorem 6.4** (finite-support obstruction at $1/21$)**.** *For every finite set $F$ of exponents with $n \geq 2$ for all $n \in F$,*

$$\sum_{n \in F} \frac{1}{2^n - 1} \neq \frac{1}{21}.$$

This is checked by the formal theorem. The reduced denominator 21 has doubling order 6, so the exact denominator–lcm identity forces every selected exponent to divide 6. The lower bound $n \geq 2$ leaves only $2, 3, 6$, and the remaining eight subsets are discharged by exact rational arithmetic. Thus any representation of $1/21$ along this rank-$\geq 2$ route, if one exists, must be infinite. The theorem proves nontermination only: it does not prove that $1/21$ lies in $\mathcal{A}$.

A separate arithmetic module isolates the primitive cone that appears in one candidate expansion route. Write

$$\mathcal{D}_{23}(n) = \{(p, q) \in \mathbb{N}_{>0}^2 : \gcd(p, q) = 1,\ 2p + 3q = n\}.$$

Every $n \geq 11$ has a member of $\mathcal{D}_{23}(n)$ (checked), whereas $\mathcal{D}_{23}(10)$ is empty (checked). Multiplicity begins immediately at 11, and every $10k$ with $k \geq 2$ has at least two distinct primitive representations (rank eleven, multiples of ten). These are exact Diophantine facts, not a counterexample construction. In particular the recurring collisions show why primitive-cone coverage is not already a Boolean expansion: the missing step is a proof that the relevant integer multiplicities can be converted into zero–one reciprocal-Mersenne digits without changing the value.

### 6.4 The canonical $1/21$ frontier

The finite-support obstruction makes $1/21$ useful only if the infinite problem is stated in coordinates that retain the actual greedy orbit. Let

$$r_N = \text{greedyMersenneRemainder}(1/21, N)$$

be the real remainder after rank $N$, and let

$$Q_N = \left\lfloor \frac{2^N}{21} \right\rfloor - P_N,$$

where $P_N$ is the integral numerator of the binary divisor-incidence prefix generated by that same greedy support. Thus $Q_N$ is the nonnegative denominator-21 defect after the six-periodic residue of $2^N$ mod 21 has been removed. These are not independent models: the formal source proves an exact identity relating $Q_N$, $2^N r_N$ and a finite-prefix divisor tail (checked).

The sharper route works directly with the denominator-21 greedy orbit. At even depth $2R$, write

$$D_R = \text{twentyOneEvenQuotientGreedySupport}(R),$$
$$s_R = \text{twentyOneEvenQuotientGreedyRemainder}(R).$$

Here $D_R \subseteq \{2, \dots, R\}$ is the Boolean support obtained by descending integer-greedy selection on the exact scaled quotient weights, and $s_R$ is its terminal scalar remainder (support, remainder). Write $\mathcal{F}_{21}$ for `TwentyOneFatalAlignedBranch`, the explicit branch in which the real greedy orbit has a fatal witness, only finitely many skipped exponents (hence cofinite eventual selection), eventual quotient/rational-greedy alignment, and eventual occupation of every doubling block.

**Theorem 6.5** (canonical quotient-greedy frontier at $1/21$)**.** *The following statements hold.*

1. $1/21 \in \mathcal{A}$ *if and only if* $\mathcal{F}_{21}$ *does not hold.*
2. *If there is an unbounded sequence of ranks* $R$ *with* $s_R \le 2^R$*, then* $1/21 \in \mathcal{A}$*.*
3. *On* $\mathcal{F}_{21}$*, eventually* $s_R > 2^R$*, the boundary rank* $R+1$ *belongs to* $D_{R+1}$*, and* $(D_R, s_R)$ *follows one exact affine recurrence.*

The equivalence is kernel checked; the closed-row compactness step is kernel checked; and the eventual affine regime is kernel checked. Explicitly, on $\mathcal{F}_{21}$ the quotient orbit is eventually strictly supercapacity and loses its last Boolean choice: if $\tau_R$ is the periodic target pulse and $\pi_R$ the divisor pulse generated by $D_R$, then for every sufficiently large $R$,

$$s_R > 2^R, \qquad D_{R+1} = D_R \cup \{R+1\}, \qquad s_{R+1} = 4s_R + \tau_R - \pi_R - (2^{R+1} + 1).$$

The denominator-specific separation theorem additionally proves that every closed Boolean quotient row is exactly the canonical quotient-greedy row, including exact saturation at the boundary (kernel checked). An aligned crossing from saturation into strict

supercapacity forces a missing canonical ancestor at one-third or two-thirds scale and a real skipped exponent with square-root-bounded defect (ancestor hole, scaled skip).

These conclusions remove alternative late Boolean branches; they do not exclude the full fatal/cofinite/aligned branch. On that branch the permanent affine-supercapacity recurrence is forced, so a contradiction of the recurrence would be sufficient, but the recurrence alone is not the exact membership endpoint. It is conditional, not a contradiction: synthetic pulses can sustain such an affine recurrence without being the actual divisor pulse.

## 7 Open problems

Problem 1.1 remains the frame: the settled families in Section 4 do not approach a universal quantifier, and the forced conditions in Section 3 are not known to be contradictory. For the base-2 universal problem, three endpoint questions organise the remaining discussion. The universal problem and the half-value question are not independent, and the relation between them is asymmetric. No finite support has value $1/2$ (Theorem 6.3), so $1/2 \in \mathcal{A}$ would exhibit an *infinite* support with a rational value and thereby refute Problem 1.1; equivalently, a positive answer to Problem 1.1 puts $1/2$ outside $\mathcal{A}$. The converse direction does not follow: $1/2 \notin \mathcal{A}$ would give a finite fatal-gap witness and eliminate this candidate, but would not imply the universal statement. The half-value question is therefore a one-sided test of #257, not a second problem beside it.

**Problem 7.1** (membership of $1/21$ in the Mersenne achievement set)**.** Exclude the explicit fatal/cofinite/aligned branch $\mathcal{F}_{21}$. It is sufficient either to contradict the eventual permanent affine-supercapacity recurrence forced by that branch or to force an unbounded sequence of closed canonical quotient rows; neither sufficient route is claimed to be equivalent by itself.

1. **Problem 1.1 itself**, for arbitrary infinite $A$. Section 4 is a list of families and does not approach a universal statement; Section 3 constrains every hypothetical counterexample without excluding one. Every counterexample would have unbounded scaled-tail states, sublogarithmic divisor-coverage gaps and an admissible Boolean–Möbius scaled-tail sequence; Theorem 3.3 adds its lower bound only under its stated convergence and odd-denominator hypotheses. No contradiction among the applicable constraints is known.

2. **Membership of $1/2$ in $\mathcal{A}$.** Theorem 6.3 makes this exactly equivalent to infinitely many greedy skips, and to a fatal gap on the other side; neither is proved. A proof of non-membership would take the form of one finite fatal gap.

3. **Problem 7.1.** Theorem 6.4 rules out finite support, while Theorem 6.5 reduces membership exactly to excluding $\mathcal{F}_{21}$. Contradicting its forced eventual affine-supercapacity recurrence or producing arbitrarily deep closed canonical rows would suffice. Neither input is known, and neither is promoted to an equivalence.

The half-value question remains valid on both sides; the frontier at $1/21$ is at present the sharper of the two. The questions below refine that frontier and the arithmetic interfaces around it, and are ranked by how directly an answer would move the checked boundary.

**Problem 7.2** (permanent affine supercapacity)**.** Can the *actual* denominator-21 greedy orbit satisfy $\mathcal{F}_{21}$ indefinitely? Equivalently, can the complete fatal, cofinite-selection, alignment and doubling-block hypotheses coexist with the eventual recurrence

$$s_R > 2^R, \quad D_{R+1} = D_R \cup \{R+1\}, \quad s_{R+1} = 4s_R + \tau_R - \pi_R - (2^{R+1}+1),$$

or must an arbitrarily deep closed return $s_R \le 2^R$ occur?

A contradiction from a Lyapunov function, a 2-adic obstruction, a divisibility theorem or a recurrence classification would prove $1/21 \in \mathcal{A}$ and, by Theorem 6.4, produce an infinite rational support. Conversely, an actual construction satisfying *all* clauses of $\mathcal{F}_{21}$ would prove $1/21 \notin \mathcal{A}$. Constructing only an abstract affine orbit with adversarial pulses does neither.

**Problem 7.3** (weakest native recurrence criterion)**.** Does the scaled actual greedy remainder return cofinally to one bounded interval?

$$\exists B < \infty \ \forall K \ \exists N \ge K : \qquad 2^N r_N \le B.$$

This is not merely sufficient: it is equivalent to $1/21 \in \mathcal{A}$ (checked). It asks for neither convergence, a prescribed $B$, a global bound nor bounded return gaps. Two concrete stronger targets are cofinal recurrence of $Q_N \le 1$ (consumer) and arbitrarily deep closed quotient rows $s_R \le 2^R$. Exact computation through rank 200,000 records 96 zero-defect returns and 4,956 returns to $Q_N \le 1$, with last observed ranks 193,690 and 199,930 and maximum observed gap 492; these finite data do not prove cofinality.

**Problem 7.4** (actual-orbit invariant)**.** Is there a finite-memory, 2-adic or discrepancy invariant, using the correlated divisor pulses of the actual support, that forces a closed return or forbids permanent supercapacity? More precisely, can one use a bounded window of $R$ mod 6, residues of $s_R$, endpoint divisor counts and the finite set of eventual skips to force descent or a forbidden state? Alternatively, can one prove that no bounded-memory invariant distinguishes the true orbit from synthetic permanent-supercapacity controls?

Pointwise pulse bounds are known to be too weak. The exact six-step recurrence is

$$Q_{N+6} = 64Q_N + 3(2^N \bmod 21) - L_N,$$

with $L_N$ the actual weighted repair load (checked). The formerly plausible translation-invariant six-step contraction first fails at $N = 73$; the surviving statement is the slope-aware repair-load iff at the checked frontier.

**Problem 7.5** (final-skip Diophantine exclusion)**.** Let $E = \sum_{n\ge1}(2^n-1)^{-1}$. If $1/21 \notin \mathcal{A}$, let $M$ be the last skipped exponent, $S_M$ its finite skipped prefix, and

$$a_M = \frac{1}{21} + \sum_{d\in S_M} \frac{1}{2^d - 1}.$$

Can the complete final-skip signatures be used to prove $|E - a_M| \ge \mathrm{gap}_M$, contradicting

$$0 < a_M - E < \mathrm{gap}_M\,?$$

The fatal interval is checked as an exact one-sided approximation. The reduced skipped-prefix denominator has binary order equal to the lcm of the skipped exponents and hence at least $M$ (order identity, lower bound). Parity-sensitive gcd restrictions and adjacent-numerator product inequalities are also checked, but they are necessary signatures rather than the missing upper-height or irrationality-measure estimate.

**Problem 7.6** (classification of rational targets)**.** For $L \geq 1$, classify the reduced rationals

$$\mathcal{J}_L = \left\{ \frac{p}{q} \in (0,1) : \begin{array}{l} \gcd(p,q) = 1,\ q \text{ odd},\ \operatorname{ord}_q(2) \leq L, \\ p/q \text{ has no finite Mersenne representation} \end{array} \right\}$$

for which non-membership reduces to an eventually deterministic finite-memory or affine quotient recurrence. In particular, classify $\mathcal{J}_6$ by target-pulse alphabet, finite-support obstruction, weakest membership consumer, surviving branch complexity and available Diophantine interface. Is $1/21$ minimal or unique under explicit criteria, or does another target have a strictly simpler fatal branch?

**Problem 7.7** (realised finite denominators)**.** For fixed $b \geq 2$ and $L \geq 2$, let $\mathcal{D}_b(L)$ be the reduced denominators of the nonempty finite sums $\sum_{n \in F} (b^n - 1)^{-1}$ with $\operatorname{lcm}(F) = L$. Is

$$\mathcal{D}_b(L) = \{D : D \mid b^L - 1,\ \operatorname{ord}_D(b) = L\}?$$

If not, what cyclotomic, valuation or cancellation conditions characterise the realised denominators?

The order theorem supplies the inclusion from left to right, not its converse. A counterexample and corrected classification, or effective extremal bounds within $\mathcal{D}_b(L)$, would strengthen the lead theorem and feed height information back into Problem 7.5. For scope, all squarefree values at power-of-two bases, jointly in each finite family, are settled by Corollary 1.2 and Example 1.1 of [2, author-preprint p. 4] (Corollary 5.2). Their cited corollary does not cover bases that are not powers of 2, and this note makes no global open-status or priority claim for those values.

## Statements and declarations

This manuscript is authored exposition, not proof authority. The linked Lean snapshot is authoritative only for its exact propositions; kernel checking establishes that a proposition was proved, not that it is interesting, novel, or sufficient. The displayed proof of Corollary 5.3, the derivation of Corollary 5.2 from [2, Cor. 1.2 and Ex. 1.1, author-preprint p. 4], and the general finite-change argument (from the two checked prefix lemmas) are expository arguments rather than named checked statements. The squarefree no-go interfaces, shifted coefficient, shift equivalence, Chinese-remainder block supply, and restricted-selector injectivity are directly checked statements linked at their use; they are not prose-only claims. The numerical instances — the table of finite supports in Section 2, the incidence and zero-window examples, the reciprocal-mass and Boolean–Möbius instances, the $3/4$ certificate, and the greedy expansion of $1/2$ through level 21 — are direct computations, reported as computations and not as

theorems. Results attributed to Campbell, to Duverney and Tachiya, to Tao and Teräväinen, to Luca and Tachiya, and to Kovač and Tao are cited from the literature and are not formalised here.

*Scope of the AI declaration.* The declaration on the first page applies in full: large-language-model agents assisted throughout, drafting and revising this exposition, the Lean developments it links to, and the repository software that checks them. Will Cook set the objectives, selected and reviewed the public claims, checked the cited literature, and authorised this manuscript, and is responsible for its contents. Formal authority is the pinned kernel's acceptance of an exact proposition; no model output carries any, and neither does this sentence.

Erdős Problem #257 remains open.

## References

[1] J. M. Campbell, *On the binary digits of the Erdős–Borwein constant*, arXiv:2605.24160v1, 2026. Theorem 1 on p. 12 proves that the binary block 11 occurs infinitely often in the base-2 expansion of the full-support value; its proof occupies pp. 12–24.

[2] D. Duverney and Y. Tachiya, *Refinement of the Chowla–Erdős method and linear independence of certain Lambert series*, Forum Math. 31 (2019), no. 6, 1557–1566, DOI. Corollary 1.2 gives the general $F_s(E)$ theorem and its monomial images under $|q|^{\mathrm{lcm}(1,\dots,\ell)} \le s$; Example 1.1 gives the joint squarefree family at all bases $2^j$, $j \ge 1$. Both are on p. 4 of the linked author preprint; the proof is on pp. 9–11.

[3] T. Tao and J. Teräväinen, *Quantitative correlations and some problems on prime factors of consecutive integers*, arXiv:2512.01739 (submitted December 2025, revised April 2026). Theorem 1.3 proves $\sum_{n\ge1} \omega(n)/2^n = \sum_p (2^p-1)^{-1}$ irrational, settling the prime-support case of #257 at base 2; the extension to every integer base and the prime-power support are stated as remarks with the modifications left to the reader. The theorem is on p. 4 and its proof is Section 5, pp. 44–56, in arXiv v2.

[4] F. Luca and Y. Tachiya, *Linear independence results for the values of divisor functions series*, RIMS Kôkyûroku No. 2014 (2017), 138–150. Theorem A on p. 139 explicitly restates their earlier Theorem 1.1 for nonzero purely periodic integer weights. Theorem 1 is on p. 139, its examples are on p. 140, and its proof is on pp. 149–150.

[5] V. Kovač and T. Tao, *On several irrationality problems for Ahmes series*, Acta Math. Hungar. 175 (2025), 572–608, DOI. Remark 4.1 (p. 13) records the strict tail inequality and the Cantor structure. Theorem 2.3 (p. 5; proof pp. 13–14) constructs rational merged sums from several bases under its mass hypothesis; it is not a fixed-base counterexample.

[6] P. Erdős, *On arithmetical properties of Lambert series*, J. Indian Math. Soc. 12 (1948), 63–66.

[7] P. Erdős, *On the irrationality of certain series*, Math. Student 36 (1968), 222–226 (issued 1969). The theorem on p. 222 treats pairwise-coprime support with convergent reciprocal sum at every integer base $b \geq 2$; the claimed removal of pairwise coprimality is stated without proof.

[8] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monogr. Enseign. Math. 28, Geneva, 1980, p. 62.

[9] P. Erdős, *On the irrationality of certain series: problems and results*, in A. Baker (ed.), *New Advances in Transcendence Theory*, Cambridge UP, 1988, pp. 102–109, doi:10.1017/CBO9780511897184.009.

[10] T. F. Bloom, *Erdős Problem #257*, `erdosproblems.com/257`, accessed 28 July 2026 (page displays "last edited 15 April 2026"). The current record labels the universal fixed-base problem open, cites `[Er68d]`, `[ErGr80, p. 62]` and `[Er88c, p. 105]`, records the Tao–Teräväinen prime and prime-power cases and the Kovač–Tao refutation of a broader varying-denominator speculation, and explicitly describes its status as the website owner's present assessment rather than a literature-completeness guarantee.