An Integral-Shift Criterion for Dyadic Tail Recurrences

Erdős Problem #251, with the principal statements checked in Lean 4

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Abstract

Call a sequence $(T_N)_{N\geq 0}$ of rational numbers a *dyadic tail recurrence with digits* g when $T_{N+1} = 2T_N - g_{N+1}$ for every N, with integer digits g_N. Its block identity is

$$T_{N+h} = 2^h T_N - (g_{N+1}2^{h-1} + \cdots + g_{N+h}).$$

Thus $T_{N+h} - T_N$ is integral exactly when $(2^h - 1)T_N$ is. Integrality of one shift propagates forward, and Euler's congruence shows that a rational term with odd reduced denominator d has an integral shift of length $\varphi(d)$. More precisely, for a real sequence obeying the same recurrence with integer digits, the initial state is rational if and only if one positive-length shift is integral at one index, equivalently if and only if one positive shift is eventually integral. Dually, the initial state is irrational exactly when every positive shift is non-integral at every index, equivalently when every fixed shift is non-integral cofinally. This abstract criterion is proved; the required cofinal property for the actual prime-gap tails is not.

Let $p_0 = 2, p_1 = 3, \ldots$ enumerate the primes and let $g_n = p_{n+1} - p_n$. The polynomial bound $p_n \leq 1250(n+1)^4$ gives summability of the prime and gap terms. Summation by parts, with the endpoint retained, yields

$$\sum_{i=0}^{n} \frac{p_i}{2^{i+1}} \;=\; 2 + \sum_{i=0}^{n-1} \frac{g_i}{2^{i+1}} - \frac{p_n}{2^{n+1}},$$

and therefore the unconditional identity and irrationality equivalence

$$\sum_{i\geq 0} \frac{p_i}{2^{i+1}} = 2 + \sum_{i\geq 0} \frac{g_i}{2^{i+1}}, \qquad \mathrm{Irr}\left(\sum_{i\geq 0} \frac{p_i}{2^{i+1}}\right) \Longleftrightarrow \mathrm{Irr}\left(\sum_{i\geq 0} \frac{g_i}{2^{i+1}}\right).$$

Problem #251 remains open. Under non-irrationality, the denominator decomposition selects a positive fixed shift that is eventually integral but not eventually confined to $(-1, 1)$. This is compatible with rationality, not a contradiction. A contradiction would require eventual smallness for that shift, or cofinally many adjacent small shifts at indices where the prime gaps differ; neither is proved. Unboundedness and nonperiodicity of the prime gaps, although both Lean-checked, do not supply either property. Indeed, an explicit unbounded, non-eventually-periodic integer sequence has a rational dyadic sum, so the missing argument must control the full prime-gap tails rather than the coefficients alone.

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1 Introduction

Let p_n denote the nth prime. Erdős Problem #251 asks whether

$$\Pi = \sum_{n\geq 1} \frac{p_n}{2^n}$$

is irrational [1, p. 94][2, p. 62][4, p. 103]. Bloom's current catalogue record reproduces this question and labels it open, while explicitly warning that the status is the website owner's present assessment and may omit relevant literature [13]. We therefore use the catalogue for numbering and current reported status only; the three original publications carry the mathematical claims. The problem is open. Numerically $\Pi = 2/2 + 3/4 + 5/8 + 7/16 + 11/32 + \cdots = 3.674643966\ldots$.

Throughout the formal development the primes are indexed from zero, so that $p_0^{(0)} = 2$, $p_1^{(0)} = 3$, $p_2^{(0)} = 5$, and the gaps are $g_i = p_{i+1}^{(0)} - p_i^{(0)}$. In that indexing

$$\Pi = \sum_{i\geq 0} \frac{p_i^{(0)}}{2^{i+1}},$$

which is the convention every statement below uses; we drop the superscript from here on. The first values are

i	0	1	2	3	4	5	6	7	8	9
p_i	2	3	5	7	11	13	17	19	23	29
g_i	1	2	2	4	2	4	2	4	6	2

so $g_0 = 1$ and every later gap is even, the primes after 2 being odd. A second normalisation with denominator 2^i also occurs, and at every finite horizon it is exactly twice this one,

$$\sum_{i=0}^{n-1} \frac{p_i}{2^i} = 2\sum_{i=0}^{n-1} \frac{p_i}{2^{i+1}},$$

the factor-of-two normalisation. A factor of two does not change rationality, so no result below depends on the choice; the identity is stated so that the indexing cannot drift silently.

Notation. $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{>0} = \{1, 2, \ldots\}$. We write φ for Euler's totient function, $\operatorname{den} q$ for the denominator of a rational number q written in lowest terms, $\operatorname{dist}(x, \mathbb{Z})$ for the distance from a real number x to the nearest integer, and $\operatorname{Irr}(x)$ for the assertion that x is irrational. A rational number is called *integral* when it is the image of an integer. A sequence (a_n) is *eventually periodic with period* $h \geq 1$ when $a_{n+h} = a_n$ for every sufficiently large n. A sum over an empty range of indices is zero.

Relation to prior work. Erdős proved that $\sum_n p_n^k/n!$ is irrational for every $k \geq 1$ [1, p. 93]. The 1958 article gives the $k = 1$ proof on pp. 94–95 and says explicitly that the more complicated $k > 1$ proof is omitted. On page 103 of his 1988 problem paper he separately stated the fixed-denominator problem: he could not prove that $\sum_n p_n^k/2^n$ is irrational for every $k \geq 1$, and wrote that the case $k = 1$ was probably already very

difficult. He also stated the variable-denominator expectation that $\sum_{n\geq 1} p_n/(g_1 \cdots g_n)$ is irrational whenever $g_n \geq 2$ and $g_n = o(p_n)$ [4, p. 103]. The latter expectation is false: ChatGPT 5.4 Pro, orchestrated by Vjeko Kovač, constructs such a sequence (g_n) for which the sum is exactly 1 [14, Theorem 1 and proof, pp. 1–2]. Under the stronger hypotheses that (g_n) is strictly increasing and $g_n = O(n \log^k n)$, Erdős had already classified the rational cases: rationality holds exactly when $g_n = qp_n + 1$ eventually for one fixed integer $q \geq 1$ [1, pp. 96–99]. The endpoint $q = 1$ is included: if $G_n = \prod_{j=1}^{n}(p_j + 1)$, then $p_n/G_n = 1/G_{n-1} - 1/G_n$, so the corresponding series telescopes to 1. The counterexample evades those hypotheses. It does not address the fixed-denominator dyadic series studied here. The current catalogue record still repeats this variable-denominator expectation without mentioning the counterexample [13]; its warning about possibly missing literature is therefore material here, not merely boilerplate.

There is also a genuinely adjacent proved dyadic theorem. If $P(m)$ denotes the largest prime factor of m, Erdős and Pomerance proved that

$$\sum_{m\geq 2} \frac{\mathbf{1}_{\{P(m)>P(m+1)\}}}{2^m}$$

is irrational [3, §7, p. 320]. Erdős and Graham record the complementary indicator on p. 62: equality of the two largest prime factors is impossible for consecutive integers, so its series is $1/2$ minus the displayed one and is irrational as well. This is a theorem about a bounded prime-factor comparison digit sequence, not the unbounded prime numerators in Π; it supplies nearby positive evidence without solving Problem #251.

Problem #251 also appears as the unproved declaration `erdos_251` in the *Formal Conjectures* repository [15]. Its zero-based Lean sum starts with the zeroth prime over 2^0, so it is twice the displayed normalisation and has equivalent irrationality status, but it is not literally the same indexing; its proof is `sorry`. No priority is claimed for anything below.

The strategy. The digits p_n grow, so the series is not a digit expansion in any bounded alphabet, and the standard rationality criteria for such expansions do not apply directly. The classical elementary criteria for series of this kind instead control irrationality through the growth of the denominators. Erdős and Straus named a sum $\sum_k 1/a_k$ over a strictly increasing sequence of positive integers an *Ahmes series* [5]; for such a series the condition $a_k^{1/2^k} \to \infty$ is sufficient for irrationality, and it is sharp, since shifted Sylvester sequences grow like C^{2^k} for arbitrarily large C and have rational reciprocal sum. Both statements and their attribution are recorded in the introduction of Kovač and Tao [7, §1], who develop the elementary technology for such series much further. Splitting each term $p_i/2^{i+1}$ into p_i copies of $2^{-(i+1)}$ writes Π as a sum of unit fractions, but with repetitions, and the denominators occurring in it are exactly the powers of two. Even ignoring the repetitions the growth hypothesis fails by every available margin, since $(2^n)^{1/2^n} \to 1$, and every arithmetic constraint has to come from the numerators instead.

What replaces growth control is denominator control on the sequence of rescaled tails. Rationality of the sum turns out to be equivalent to an eventual integrality condition on differences of those tails, and that condition uses nothing about the numerators

beyond the fact that they are integers. The condition does not by itself force the numerators to repeat: Proposition 4.1, applied to $K_n = n$, produces the integer sequence $\kappa_n = n - 1$, which is unbounded and hence not eventually periodic, and whose dyadic sum $\sum_{n\geq 0} \kappa_n 2^{-(n+1)}$ is zero.

Outline. Section 2 replaces the primes by their consecutive gaps, the natural increments studied by prime-distribution theory. Summation by parts with the endpoint retained gives an exact finite identity, and the termwise relation $v_n = 2u_{n+1} - u_n$ between the gap terms and the prime terms makes the passage to the limit a matter of summability alone; the explicit polynomial prime bound supplies that summability inside the formal development. Section 3 rescales the tails of a dyadic series into a recurrence and develops it: the block identity, the integral-shift criterion, the collapse of every rational solution onto an eventually integral shift, its contrapositive over the reals, and a local obstruction that converts an infinite-tail condition into a single comparison of gaps. Section 4 shows that the tail constraint does not by itself make the coefficient sequence eventually periodic, and Section 5 states the remaining obligation. The pinned Lean 4 toolchain and Mathlib revision check the formal statements. The cited system paper identifies Lean 4 [8, abstract and §1, pp. 625–626], while the Mathlib paper documents the library's historical Lean 3-era architecture [9, abstract and §1.1, p. 367]; it is not authority for the current pinned revision. Linked phrases open the corresponding declaration at the pinned source revision c8e41c76b4ce.

Status. The problem treated here is open, and this note does not close it. Every statement below marked as checked is a proposition that the pinned Lean kernel accepts from the sources this note links to, with no `sorry`, no added axiom, and no unchecked evaluation. That is a claim about the formal statement, not about its mathematical interest, its novelty, or the original problem. The unresolved obligations are named exactly, in their own section, and none of the finite computations, reductions, or no-go results here removes one of them.

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2 Summation by parts, with the endpoint retained

Summation by parts trades a sequence for its consecutive differences. The form recorded here is exact: it carries no error term, it assumes nothing about the sequence, and it retains the endpoint term rather than absorbing it into an estimate. For a sequence P of rational numbers and $n \geq 0$ put

$$D(P,n) = \sum_{i=0}^{n-1} \frac{P(i)}{2^{i+1}}, \qquad \Delta(P,n) = \sum_{i=0}^{n-1} \frac{P(i+1) - P(i)}{2^{i+1}},$$

both empty, hence zero, at $n = 0$. We call these the dyadic partial sum and the dyadic difference sum of P.

Proposition 2.1 (finite summation by parts)**.** *For every rational sequence P and every $n \geq 0$,*

$$D(P,n+1) = P(0) + \Delta(P,n) - \frac{P(n)}{2^{n+1}}.$$

Statement	Status	Treatment here
Irrationality of Π	Open	Not proved.
Finite prime-gap identity	Proved here	Theorem 2.2, with the endpoint retained.
Infinite prime-gap identity	Lean-checked unconditionally	Theorem 2.3, with summability discharged by the polynomial prime bound.
Prime-series/gap-series irrationality equivalence	Lean-checked unconditionally	Corollary 2.4.
Block identity for the tail recurrence	Proved here	Theorem 3.2.
Integral shift criterion	Proved here; an equivalence	Theorem 3.3.
Totient shift from an odd denominator	Proved here	Theorem 3.4.
Propagation of an integral shift	Proved here	Theorem 3.5.
Eventual integral shift for every rational-valued recurrence	Lean-checked	Theorem 3.6.
Rationality/integral-shift classification	Lean-checked abstractly; an equivalence	Theorem 3.7.
Finite-approximation gap	Paper-level	Elementary inference used in Proposition 5.4; not a named Lean declaration at the pinned revision.
Actual prime gaps are unbounded and not eventually periodic	Lean-checked	Proposition 3.10.
Adjacent small-mismatch pair excludes simultaneous integrality	Lean-checked	Theorem 3.8.
Concrete prime-gap tail recurrence and rational-candidate bridge	Paper-level recurrence; Lean-checked conditional bridge	The rational candidate recurrence is unconditional; its representation of the actual scaled tail assumes non-irrationality.
Rationality alone forces periodic integer coefficients	False	Section 4, Proposition 4.1.
Cofinal adjacent small-mismatch hypothesis	Proposed sufficient theorem	Problem 5.3; not proved.

How to read the middle column. *Proved here* marks a statement proved in the text; each such statement also carries a link to a Lean declaration where it appears below. *Lean-checked* marks a statement the pinned kernel accepts, in the exact sense fixed by the Status paragraph above. The modifier *abstractly* marks a statement proved for an arbitrary integer digit sequence rather than for the actual prime gaps. *Proposed sufficient theorem* marks an unproved statement which, if proved, would give irrationality of Π.

Proof. A routine induction on n. At $n = 0$ both sides equal $P(0)/2$. For the step, adding $P(n+1)/2^{n+2}$ to the left and $(P(n+1) - P(n))/2^{n+1}$ to the difference sum changes the endpoint term from $P(n)/2^{n+1}$ to $P(n+1)/2^{n+2}$, and the two adjustments agree. □

Formalised as the summation-by-parts identity. Nothing is assumed about P: no positivity, no monotonicity, and no convergence.

Specialising to $P(i) = p_i$, whose first value is $p_0 = 2$, and writing $g_i = p_{i+1} - p_i$ for the zero-based gaps, gives the reformulation.

Theorem 2.2 (prime-gap reformulation)**.** *Let $p_0 = 2, p_1 = 3, \ldots$ be the primes in increasing order and $g_i = p_{i+1} - p_i$. For every $n \geq 0$,*

$$\sum_{i=0}^{n} \frac{p_i}{2^{i+1}} = 2 + \sum_{i=0}^{n-1} \frac{g_i}{2^{i+1}} - \frac{p_n}{2^{n+1}}.$$

Formalised as the prime-gap summation by parts, using the gap partial sum and the gap cast identity; the latter records that the natural-number difference $p_{n+1} - p_n$ agrees with the difference taken in $\mathbb{Q}$, which needs $p_n \leq p_{n+1}$, the monotonicity step. The leading 2 is the first prime, not a normalising constant. At $n = 2$, for instance, the left side is $2/2 + 3/4 + 5/8 = 19/8$ and the right side is $2 + (1/2 + 2/4) - 5/8 = 19/8$.

2.1 The infinite identity and the irrationality equivalence

Write

$$u_n = \frac{p_n}{2^{n+1}}, \qquad v_n = \frac{g_n}{2^{n+1}}$$

for the terms of the prime series and of the gap series, so that $\sum_{n\geq 0} u_n = \Pi$. The termwise identity $v_n = 2u_{n+1} - u_n$ is the dyadic discrete derivative. It expresses each gap term as an integer combination of two consecutive prime terms, so once (u_n) is summable the gap series can be summed by rearranging two copies of the prime series, which is what the following proof does.

Theorem 2.3 (infinite prime-gap identity)**.** *Let $u_n = p_n/2^{n+1}$ and $v_n = g_n/2^{n+1}$. If (u_n) is summable, then (v_n) is summable and*

$$\sum_{n\geq 0} u_n = 2 + \sum_{n\geq 0} v_n.$$

Proof. The shifted sequence (u_{n+1}) is summable. Sum $v_n = 2u_{n+1} - u_n$ and use $u_0 = 1$:

$$\sum_{n\geq 0} v_n = 2\left(\sum_{n\geq 0} u_n - u_0\right) - \sum_{n\geq 0} u_n = \sum_{n\geq 0} u_n - 2.$$

□

The summability transfer is the gap-series summability theorem, and the displayed identity is the infinite prime-gap identity.

Corollary 2.4 (exact irrationality reformulation)**.** *With u_n and v_n as in Theorem 2.3, and (u_n) summable,*

$$\mathrm{Irr}\left(\sum_{n\geq 0} u_n\right) \quad \Longleftrightarrow \quad \mathrm{Irr}\left(\sum_{n\geq 0} v_n\right).$$

The corresponding zero-based series with denominator 2^n is $4 + 2\sum_{n\geq 0} v_n$ and has the same irrationality status.

These are the normalised irrationality equivalence, the displayed-series identity, and the displayed-series irrationality equivalence.

The summability hypothesis. The formal source now proves the elementary polynomial bound $p_n \leq 1250(n+1)^4$, using prime counting and central-binomial growth, and deduces summability directly (polynomial bound, summability). Thus Theorem 2.3 and Corollary 2.4 are now unconditional Lean-checked statements; the prime number theorem remains useful context but is no longer a proof dependency of this note. The open content is exactly irrationality of the prime-gap series. Concretely, $\Pi = 3.674643966\ldots$ is irrational if and only if $\sum_{n\geq 0} g_n 2^{-(n+1)} = 1.674643966\ldots$ is, and neither is known.

3 The tail recurrence and integral shifts

The series is not attacked directly. Suppose $\sum_{i\geq 0} a_i 2^{-(i+1)}$ converges, with every a_i an integer, and rescale its tails by putting

$$T_N = 2^{N+1} \sum_{i>N} \frac{a_i}{2^{i+1}} = \sum_{j\geq 1} \frac{a_{N+j}}{2^j}.$$

Two facts follow immediately. First $T_{N+1} = 2T_N - a_{N+1}$, so moving one level along doubles the rescaled tail and subtracts a single coefficient. Second $T_0 = 2\sum_{i\geq 0} a_i 2^{-(i+1)} - a_0$, so the sum is rational exactly when T_0 is. The whole of Section 3 therefore studies that recurrence in isolation, assuming nothing about the coefficients beyond the fact that they are integers; the prime-gap instance is resumed at the end of the section. The following definition is the object so obtained, stripped of its origin.

Definition 3.1. Let $g : \mathbb{N} \to \mathbb{Z}$ and $T : \mathbb{N} \to \mathbb{Q}$. Say T satisfies the *dyadic tail recurrence* with *digits* g when

$$T_{N+1} = 2T_N - g_{N+1} \qquad \text{for every } N.$$

We call the sequence $(T_N)_{N\geq 0}$ the *orbit* of T_0 under g, write $\sigma_h(N) = T_{N+h} - T_N$ for the *shift* of length h at N, and call a rational number *integral* when it is the image of an integer.

These are the tail recurrence, the shift, and integrality. The digits are arbitrary integers. In the intended instance they are the prime gaps and T_N is the scaled tail of Π after level N, but that instantiation needs a summability argument and is not made here; every statement below is a theorem about Definition 3.1.

One small orbit, referred to again below, is worth having in view. Take every digit $g_N = 0$ and $T_0 = 1/12$. Then $T_{N+1} = 2T_N$, so the orbit is $\frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots$, and $\sigma_h(N) = (2^h - 1)2^N/12$. The shift of length 2 is not integral at $N = 0$, where $\sigma_2(0) = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$, and is integral at $N = 2$, where $\sigma_2(2) = \frac{4}{3} - \frac{1}{3} = 1$; the shift of length 1 is $\sigma_1(N) = 2^N/12$ and is never integral. The change at $N = 2$ is accounted for by Theorem 3.6, and the failure at $h = 1$ by Theorem 3.3.

Iterating the recurrence h times multiplies T_N by 2^h and accumulates an explicit integer, which we now name. Define $B_{0,N} = 0$ and $B_{h+1,N} = 2B_{h,N} + g_{N+h+1}$, so that $B_{h,N} = g_{N+1}2^{h-1} + \cdots + g_{N+h}$: the tail block. Thus $B_{1,N} = g_{N+1}$, $B_{2,N} = 2g_{N+1} + g_{N+2}$ and $B_{3,N} = 4g_{N+1} + 2g_{N+2} + g_{N+3}$: the block puts the weights $2^{h-1}, \ldots, 2^0$ on the h digits following index N.

Theorem 3.2 (block identity)**.** *Let $T : \mathbb{N} \to \mathbb{Q}$ satisfy the dyadic tail recurrence with integer digits g, and let $B_{h,N}$ be as above. For every N and h,*

$$T_{N+h} = 2^h T_N - B_{h,N}, \qquad \textit{hence} \qquad \sigma_h(N) = (2^h - 1)\, T_N - B_{h,N}.$$

Proof. A routine induction on h; the step is one application of the recurrence together with $2 \cdot 2^h = 2^{h+1}$ and the recursion defining B. The second identity is the first minus T_N. □

Formalised as the iterated block identity and the scaled shift identity. The shift also obeys the recurrence in its own right, $\sigma_h(N+1) = 2\sigma_h(N) - (g_{N+h+1} - g_{N+1})$, the shift step identity.

Since $B_{h,N}$ is an integer, Theorem 3.2 converts a question about the shift into a question about the single scaled term $(2^h - 1)T_N$.

Theorem 3.3 (integral-shift criterion)**.** *Let $T : \mathbb{N} \to \mathbb{Q}$ satisfy the dyadic tail recurrence with integer digits g. For every N and h, the shift $\sigma_h(N)$ is integral if and only if $(2^h - 1)T_N$ is integral.*

Proof. By Theorem 3.2 the two differ by the integer $B_{h,N}$, and subtracting an integer does not change integrality. □

Formalised as the integral-shift criterion, on the integer-shift invariance. This is an equivalence, not a one-way reduction: the block carries no information about integrality, so the two conditions are the same condition. Informally, once T_N is fixed no choice of the digits after index N can change whether the shift $\sigma_h(N)$ is an integer; that is decided by the denominator of T_N alone.

The denominator criterion is exact:

$$\sigma_h(N) \in \mathbb{Z} \quad \Longleftrightarrow \quad \operatorname{den}(T_N) \mid 2^h - 1. \tag{3.4}$$

This is the denominator classification. Euler's totient supplies one admissible shift length when the denominator is odd; it is a witness, not the classification itself.

Theorem 3.4 (a shift of totient length)**.** *Let $T : \mathbb{N} \to \mathbb{Q}$ satisfy the dyadic tail recurrence with integer digits g, and let φ be Euler's totient function. If the reduced denominator d of T_N is odd, then $\sigma_{\varphi(d)}(N)$ is integral.*

Proof. Since d is odd, 2 and d are coprime, so Euler's congruence gives $2^{\varphi(d)} \equiv 1 \pmod{d}$, that is $d \mid 2^{\varphi(d)} - 1$. Writing $2^{\varphi(d)} - 1 = dk$ and $T_N = u/d$ in lowest terms, $(2^{\varphi(d)} - 1)T_N = ku$ is an integer, and Theorem 3.3 transfers this to the shift. □

Formalised as the totient shift and the Euler multiplier. For example, if $\operatorname{den} T_N = 3$ then $\varphi(3) = 2$ and $(2^2-1)T_N = 3T_N$ is an integer, so $\sigma_2(N)$ is integral, while $(2^1-1)T_N = T_N$ is not; if $\operatorname{den} T_N = 5$ then $\varphi(5) = 4$ and $\sigma_4(N)$ is integral.

The hypothesis is a genuine restriction: the argument uses coprimality of 2 with the denominator, and the even part of a denominator is exactly what the doubling in the recurrence acts on. It cannot be dropped. If $T_N = 1/2$ then $2^h - 1$ is odd for every $h \geq 1$, so $(2^h - 1)T_N$ is never an integer and, by Theorem 3.3, no shift at N is integral.

Theorem 3.5 (propagation)**.** *Let $T : \mathbb{N} \to \mathbb{Q}$ satisfy the dyadic tail recurrence with integer digits g, and fix h and N. If $\sigma_h(N)$ is integral, then $\sigma_h(N + k)$ is integral for every $k \geq 0$.*

Proof. By the shift step identity, $\sigma_h(N + 1) = 2\sigma_h(N) - (g_{N+h+1} - g_{N+1})$ is an integer combination of an integer and two digits; induct on k. □

Formalised as the one-step propagation and the propagation to every later index.

The three preceding theorems combine as follows, and this is the statement the rest of the note rests on. The special case is immediate: if $\operatorname{den} T_0$ is already odd, then Theorem 3.4 at $N = 0$ makes the shift of length $\varphi(\operatorname{den} T_0)$ integral and Theorem 3.5 keeps it integral at every later index, so one may take $N_0 = 0$. In general a denominator carries a power of two as well, and the key point is that the doubling in the recurrence annihilates exactly the 2-adic part of a denominator, and nothing else: after finitely many steps the orbit therefore reaches a term with odd reduced denominator, which is precisely the situation Theorem 3.4 handles. No control of the digits is needed anywhere.

Theorem 3.6 (exact denominator dynamics and eventual integrality)**.** *Let $T : \mathbb{N} \to \mathbb{Q}$ satisfy the dyadic tail recurrence with integer digits g (Definition 3.1). Then some fixed positive shift is integral from some point onwards:*

$$\exists h \geq 1 \; \exists N_0 \; \forall N \geq N_0, \qquad \sigma_h(N) \in \mathbb{Z}.$$

Proof. At every step the reduced denominator obeys the exact recurrence

$$\operatorname{den}(T_{N+1}) = \frac{\operatorname{den}(T_N)}{\gcd(2, \operatorname{den}(T_N))}.$$

Thus each even denominator loses exactly one factor of 2, while an odd denominator is unchanged. After finitely many steps the denominator is odd. Theorem 3.4, applied at that index s, supplies the positive shift $h = \varphi(\operatorname{den} T_s)$, and Theorem 3.5 keeps that shift integral at every later index. □

The one-step formula is the denominator recurrence, with its odd case and even case.

In the orbit displayed after Definition 3.1 the proof runs as follows: $\operatorname{den} T_0 = 12 = 2^2 \cdot 3$, so $s = 2$, the orbit reaches $T_2 = 1/3$ with odd denominator, and $h = \varphi(3) = 2$. That is exactly the shift length seen to be integral there from index 2 onwards, and no shorter one works.

Three features of the argument are used later. The proof may take as its number of preparatory steps the 2-adic valuation of den T_0. The resulting shift length h is the totient of the odd denominator reached from a hypothetical rational initial value, and is not known in advance for the prime-gap orbit. This is why this argument requires Problem 5.2 for every h, rather than for one preassigned shift length. Finally, the digits enter only through the integer $B_{s,0}$, so the conclusion holds for an arbitrary integer digit sequence.
In the current formal source, denominator factorisation and cancellation are packaged directly in the fixed-denominator theorem, and the quantified conclusion is the eventual integral-shift theorem for the actual prime-gap tail state.

The useful contrapositive is stated for a real recurrence. Call its shifts *cofinally non-integral* when, for every fixed $h \geq 1$ and every threshold N_0, some $N \geq N_0$ has $\sigma_h(N) \notin \mathbb{Z}$. This is precisely the negation of the conclusion of Theorem 3.6: no shift length whatever becomes integral and stays integral.

Theorem 3.7 (exact rationality classification)**.** *Let $T : \mathbb{N} \to \mathbb{R}$ satisfy $T_{N+1} = 2T_N - g_{N+1}$ with integer g. If its shifts are defined by $\sigma_h(N) = T_{N+h} - T_N$, then the following are equivalent:*

(i) *T_0 is rational;*

(ii) *$\sigma_h(N)$ is integral for some $h \geq 1$ and some N;*

(iii) *for some fixed $h \geq 1$, $\sigma_h(N)$ is integral at every sufficiently large N.*

Consequently T_0 is irrational if and only if every positive-length shift is non-integral at every index, equivalently if and only if the shifts are cofinally non-integral.

Proof. For (i)⇒(iii), choose $q \in \mathbb{Q}$ whose real cast is T_0. The real block identity identifies the whole orbit with the cast of the rational recurrence starting at q; Theorem 3.6 applied to that rational orbit then gives (iii). The implication (iii)⇒(ii) is immediate. For (ii)⇒(i), the real block identity gives

$$\sigma_h(N) = (2^h - 1)T_N - B_{h,N}.$$

Here $B_{h,N}$ and $\sigma_h(N)$ are integers and $2^h - 1 \neq 0$, so T_N is rational. Iterating the recurrence backwards through the block identity then makes T_0 rational. Negating the pointwise and eventual forms gives the two irrationality formulations. □

Lean checks the rational actual-tail state as rational prime-gap tail state, its recurrence, and the bridge from a hypothetical rational value to that state as the rational-tail representation. The exact real classifiers are one integral positive shift, eventual integrality, pointwise non-integrality, and cofinal non-integrality.

The actual prime-gap orbit. Define, at paper level,

$$\mathcal{T}_N = \sum_{j \geq 1} \frac{g_{N+j}}{2^j}.$$

The Lean-checked polynomial prime bound gives convergence. A paper-level index shift, justified by that convergence, gives

$$\mathcal{T}_{N+1} = 2\mathcal{T}_N - g_{N+1}.$$

If $G = \sum_{n\geq 0} g_n/2^{n+1}$, then $\mathcal{J}_0 = 2G - g_0 = 2G - 1 = 2.349287932\ldots$. Together with Theorem 2.3, this shows that Π, G, and $\mathcal{J}_0$ have the same rationality status. Thus Theorem 3.7 identifies Problem #251 exactly with cofinal shift escape for $\mathcal{J}$. Lean checks the rational candidate recurrence unconditionally and, under non-irrationality of the series, its representation of every scaled real tail. The displayed recurrence for $\mathcal{J}$ is the elementary index-shift deduction above. What is not proved is the cofinal non-integrality needed by Theorem 3.7.

3.1 Two adjacent small shifts cannot both be integral

The exact classifier reduces irrationality to cofinal non-integrality, a condition of infinite precision imposed on a complete tail. The key point is that inside the open interval $(-1, 1)$ integrality is equality with zero, so on that range the one-step shift recurrence

$$\sigma_h(N+1) = 2\sigma_h(N) - (g_{N+h+1} - g_{N+1})$$

turns simultaneous integrality of two adjacent shifts into a single comparison of digits. What this buys is a *certificate*: a condition attached to a single pair of adjacent indices, whose verification already contradicts integrality at that pair. The point of arranging the argument this way is that the distance of a shift from the integers never has to be estimated; it suffices to know that both shifts lie in $(-1, 1)$ and that two digits differ.

Theorem 3.8 (adjacent small-shift obstruction)**.** *Let $T : \mathbb{N} \to \mathbb{Q}$ satisfy the dyadic tail recurrence with integer digits g. Fix h and N. If*

$$-1 < \sigma_h(N) < 1, \qquad -1 < \sigma_h(N+1) < 1, \qquad g_{N+h+1} \neq g_{N+1},$$

then $\sigma_h(N)$ and $\sigma_h(N+1)$ cannot both be integral. Consequently, if such a pair occurs beyond every threshold, the h-shift is not eventually integral.

Proof. An integral rational strictly between -1 and 1 is zero. If both shifts were integral, both would therefore vanish, and substitution in the displayed shift step identity would give $g_{N+h+1} = g_{N+1}$, a contradiction. The cofinal statement chooses one such adjacent pair after the alleged onset of integrality. □

The finite contradiction is the adjacent small-shift obstruction; its quantified form is the cofinal small-mismatch theorem; and the actual-prime-gap specialisation is the prime-gap specialisation. Theorem 3.8 is conditional on its two tail inequalities; no theorem asserting that such pairs occur is claimed here.

All three are stated for a rational orbit, while the prime-gap tail $\mathcal{J}$ of Section 3 is real, so the rational statement does not on its own discharge the instances that arise downstream. The same argument gives the real form directly.

Corollary 3.9 (adjacent small-shift obstruction, real form)**.** *Let $T : \mathbb{N} \to \mathbb{R}$ satisfy $T_{N+1} = 2T_N - g_{N+1}$ with integer digits g, and write $\sigma_h(N) = T_{N+h} - T_N$. Fix h and N. If*

$$-1 < \sigma_h(N) < 1, \qquad -1 < \sigma_h(N+1) < 1, \qquad g_{N+h+1} \neq g_{N+1},$$

then $\sigma_h(N)$ and $\sigma_h(N+1)$ do not both lie in $\mathbb{Z}$. Consequently, if such a pair occurs beyond every threshold, the h-shift is not eventually integral.

Proof. The real recurrence gives the same shift step identity $\sigma_h(N+1) = 2\sigma_h(N) - (g_{N+h+1} - g_{N+1})$, and an integer strictly between -1 and 1 is zero. If both shifts lay in $\mathbb{Z}$, both would therefore vanish, and substitution in that identity would give $g_{N+h+1} = g_{N+1}$, a contradiction. The cofinal statement chooses one such adjacent pair after the alleged onset of integrality. □

Corollary 3.9 is proved here and is not a declaration in the pinned Lean module, whose small-shift theorems are stated for the rational orbit. It is the form that applies to the real tail, and it carries the same two unproved inequalities as its rational counterpart.

The third hypothesis, on its own, is available. For the actual gaps tabulated in Section 1 it reads $g_4 = 2 \neq 4 = g_3$ at $h = 1$, $N = 2$; and Proposition 3.10 below says precisely that for each fixed $h \geq 1$ the inequality $g_{N+h+1} \neq g_{N+1}$ holds for arbitrarily large N, since its failure from some index onwards is eventual periodicity with period h. What is missing is the pair of inequalities, each of which constrains a complete infinite tail.

Proposition 3.10 (prime gaps do not become periodic)**.** *For every positive h, the actual consecutive-prime-gap sequence is not eventually periodic with period h.*

Proof. The gaps are unbounded, by the standard construction: the interval from $n! + 2$ to $n! + n$ contains no prime. Far stronger lower bounds for large gaps are known [6, Theorem 1, p. 66], but unboundedness is all that is needed. An eventually periodic natural-valued sequence has finite range after its preperiod, while its finite initial segment is bounded as well; hence it is bounded, a contradiction. □

Lean checks the factorial argument as unboundedness of the actual gaps and the conclusion as non-eventual periodicity. Combining nonperiodicity with eventual strict smallness of one positive shift would also exclude eventual integrality, but eventual smallness at every sufficiently large index is stronger than the cofinal adjacent-pair hypothesis in Theorem 3.8 and is not asserted here.

4 Nonperiodic coefficients with a rational sum

Proposition 3.10 shows that the prime gaps are not eventually periodic. One might therefore hope that rationality of a dyadic series forces its integer coefficients to be eventually periodic, and play the two against each other. The hoped-for implication is false, and a single explicit sequence refutes it.

The construction runs the emission of coefficients backwards. Let $K : \mathbb{N} \to \mathbb{Q}$ be arbitrary, read K_n as the value carried into level n, and put $\kappa_n = 2K_n - K_{n+1}$: the carry coefficient. That definition is exactly the statement that

$$\frac{K_n}{2^n} = \frac{\kappa_n}{2^{n+1}} + \frac{K_{n+1}}{2^{n+1}},$$

so at each level the carried value splits into one emitted coefficient and a new carry. Nothing at all is assumed about K: not integrality, not positivity, not any bound. That is the sense in which the carry is free, and it is what the counterexample below exploits.

Proposition 4.1 (exact telescoping)**.** *Let $K : \mathbb{N} \to \mathbb{Q}$ be arbitrary and $\kappa_n = 2K_n - K_{n+1}$. For every $n \geq 0$,*

$$\sum_{i=0}^{n-1} \frac{\kappa_i}{2^{i+1}} = K_0 - \frac{K_n}{2^n}.$$

Proof. A routine induction on n; the added term is $(2K_n - K_{n+1})/2^{n+1} = K_n/2^n - K_{n+1}/2^{n+1}$. □

Formalised as the carry telescoping identity, on the carry partial sum. When K takes natural-number values and $K_{n+1} \leq 2K_n$ for every n, the coefficients κ_n are natural numbers as well, and the natural-number and rational readings of the definition agree under the cast, the natural-carry cast.

Consequence for the coefficients. If $K_n 2^{-n} \to 0$, the emitted partial sums converge to K_0. A parity-compatible positive example is

$$K_0 = \frac{5}{2}, \qquad K_n = 2n + 2 \ (n \geq 1), \qquad \kappa_0 = 1, \quad \kappa_n = 2n \ (n \geq 1),$$

for which

$$\sum_{i=0}^{n-1} \frac{\kappa_i}{2^{i+1}} = \frac{5}{2} - \frac{K_n}{2^n} \longrightarrow \frac{5}{2}.$$

Thus the emitted coefficients $1, 2, 4, 6, 8, 10, \ldots$ are positive, even after the first term, unbounded and not eventually periodic, although their dyadic sum is rational. Rationality alone therefore cannot imply eventual periodicity even for a positive, parity-correct integer coefficient sequence. The example does not claim that these coefficients are prime gaps; it isolates the additional arithmetic information any successful argument must use.

5 Complements and further questions

Problem #251 is open. The public Lean source now proves unconditional convergence of both dyadic series, their exact infinite summation-by-parts identity, and the real-to-rational scaled-tail representation (identity, tail representation). It also proves that every rational candidate supplies one positive fixed shift which is integral at every sufficiently late tail index (checked). The denominator construction selects that same shift so that prime-gap nonperiodicity also rules out its eventual confinement to the open unit interval (not eventually small). This conclusion is compatible with rationality: it says that the rationally forced shift cannot support an eventual-smallness contradiction. It supplies neither eventual smallness nor cofinally many adjacent small mismatches for that shift. The remaining issue is recurrence or anti-concentration of the *actual* prime-gap shifts. Throughout, $g_i = p_{i+1} - p_i$ are the zero-based prime gaps of Section 1. Write $\mathsf{E}(h)$ for the cofinal escape statement in (5.1) at shift h.

Problem 5.1 (divisor-hitting shift escape)**.** For every $r \geq 1$, does some positive multiple of r escape cofinally?

$$\forall r \geq 1 \ \exists m \geq 1 : \qquad \mathsf{E}(mr).$$

This is the weaker recurrence-level target suggested by the denominator mechanism: a hypothetical rational value produces an eventually integral fixed shift, and the tail-shift cocycle propagates integrality forward and through positive multiples of that shift. Forward propagation is Lean-checked; the short multiple-in-shift closure is an elementary paper-level derivation and has not yet been given a named declaration. Accordingly this problem is presented as a sharper proposed criterion, not as a formally registered equivalence.

Problem 5.2 (universal prime-gap shift escape)**.** For every $h \geq 1$ and every N_0, prove that some $N \geq N_0$ satisfies

$$\sum_{j\geq 1} \frac{g_{N+h+j} - g_{N+j}}{2^j} \notin \mathbb{Z}. \tag{5.1}$$

The series in (5.1) is exactly $\mathcal{T}_{N+h} - \mathcal{T}_N$. Theorem 3.7 makes Problem 5.2 equivalent to irrationality of Π. This all-shifts form is pointwise stronger than the sufficient target in Problem 5.1: a hypothetical rational value chooses a shift length from the odd part of its reduced denominator, and one needs only hit a compatible multiple rather than escape at every prescribed h.

Problem 5.3 (cofinal adjacent small mismatch)**.** For every fixed $h \geq 1$ and every N_0, prove that some $N \geq N_0$ satisfies

$$\begin{aligned} \left|\sum_{j\geq 1} \frac{g_{N+h+j} - g_{N+j}}{2^j}\right| &< 1, \\ \left|\sum_{j\geq 1} \frac{g_{N+h+1+j} - g_{N+1+j}}{2^j}\right| &< 1, \\ g_{N+h+1} &\neq g_{N+1}. \end{aligned} \tag{5.2}$$

The two sums are adjacent h-shifts. Corollary 3.9 turns each instance of (5.2) into a finite contradiction to simultaneous integrality; Theorem 3.8 is the Lean-checked rational case. A cofinal family of such pairs therefore proves Problem 5.2; Theorem 3.7 then gives irrationality of the gap series, and Corollary 2.4 transfers it to Π. This formulation deliberately asks only for sporadic adjacent pairs; the stronger assertion that a fixed shift is eventually always smaller than one is unnecessary.

Several natural prime-distribution inputs are insufficient. Isolated small gaps, isolated large gaps, average gap estimates, and the occurrence of any one fixed finite pattern do not suffice: both inequalities in (5.2) contain the complete infinite continuation. Nor does parity: after the first gap all g_n are even. Unboundedness and non-eventual-periodicity of the actual gaps, though now checked, do not imply the required small-tail recurrence. In particular, Section 4 shows that no argument can deduce eventual periodicity from rationality alone.

A finite truncation criterion. Both problems above are stated in terms of infinite tails, but a dominated truncation reduces the first of them to a finite quantity. For $L \geq 1$ put

$$S_{h,N,L} = \sum_{j=1}^{L} \frac{g_{N+h+j} - g_{N+j}}{2^j},$$

the truncation of the series in (5.1) after L terms. This is a finite sum of gap differences and can be computed; the point of the following proposition is to say how far from an integer it must be before the discarded tail is irrelevant.

Equivalently, introduce the integral dyadic block

$$D_{h,N,L} = \sum_{j=1}^{L} 2^{L-j}(g_{N+h+j} - g_{N+j}), \qquad S_{h,N,L} = \frac{D_{h,N,L}}{2^L}.$$

Then

$$\operatorname{dist}(S_{h,N,L}, \mathbb{Z}) = 2^{-L} \min\{D_{h,N,L} \bmod 2^L, 2^L - (D_{h,N,L} \bmod 2^L)\}.$$

Here $D_{h,N,L} \bmod 2^L$ denotes the least nonnegative residue, including when $D_{h,N,L} < 0$. Thus the finite criterion below is an exact modular small-arc problem: the residue of $D_{h,N,L}$ must avoid the two arcs of radius $2^L R_{h,N,L}(M)$ around 0 modulo 2^L. A one-block certificate would be a prime-gap theorem producing such an avoided arc on a logarithmic block.

Proposition 5.4 (finite truncation)**.** *Let $M(n) \geq g_n$ for every n, assume that the series below converges, and put*

$$R_{h,N,L}(M) = \sum_{j>L} \frac{M(N+h+j) + M(N+j)}{2^j}.$$

Suppose that for every fixed $h \geq 1$ and every N_0 there exist $N \geq N_0$ and $L \geq 1$ with

$$\operatorname{dist}(S_{h,N,L}, \mathbb{Z}) > R_{h,N,L}(M). \tag{5.3}$$

Then Problem 5.2 holds.

Proof. The part of $\sum_{j\geq 1}(g_{N+h+j} - g_{N+j})2^{-j}$ omitted from $S_{h,N,L}$ has absolute value at most $R_{h,N,L}(M)$, so under (5.3) the full sum lies at positive distance from every integer. □

The last distance-to-integers inference is the elementary paper argument in the displayed proof: an error at most R cannot reach an integer when the approximation is farther than R from every integer. It is not currently a named Lean declaration. For instance, if $S_{h,N,L} = 2/5$ and $R_{h,N,L}(M) = 1/20$, the full sum lies within $1/20$ of $2/5$ and so at distance at least $7/20$ from every integer, which settles (5.1) at that N. The prime-gap tail bound, the convergence used above, and the existence of blocks satisfying (5.3) are not consequences of that inference. The proposition's full sum is moreover real-valued, so the reverse-triangle step used here is a paper proof and not an instance of any rational-valued formal declaration.

Taking the classical bound $M(n) \ll n \log n$, a choice $L = \lceil A \log_2(N+h+2) \rceil$ with any fixed $A > 1$ makes the right side a negative power of N up to logarithms. Thus (5.3) asks for a finite dyadic anti-concentration estimate on a logarithmic-length block, not control of an infinite tail and not eventual periodicity of the full gap sequence. For Problem 5.3, the same truncation must certify two adjacent full-tail values inside the open unit interval, together with the displayed gap mismatch. A finite prefix is useful only when its omitted tail is rigorously dominated.

What (5.3) requires is joint control of the finite block of weighted differences $(g_{N+h+1} - g_{N+1}, \cdots, g_{N+h+L} - g_{N+L})$ modulo powers of two, along a block of logarithmic length. We do not know how to obtain such control and we make no progress on it here. The strongest results on prime gaps address a different shape of question. Zhang's bounded-gap theorem [12, Theorem 1, p. 1122] produces infinitely many bounded consecutive-prime gaps. Maynard proves substantially more than an individual-gap statement: [10, Theorem 1.1, p. 384] bounds $\liminf_n (p_{n+m} - p_n)$ for every fixed m, and hence gives bounded clusters of every fixed size; [10, Theorem 1.3, p. 385] gives the explicit unconditional bound $\liminf_n (p_{n+1} - p_n) \leq 600$. The large-gap theorem of Ford, Green, Konyagin, Maynard and Tao [6, Theorem 1, p. 66] bounds the largest single consecutive-prime gap below X. None of these results supplies the joint dyadic distribution of a logarithmic block of consecutive gap differences required by (5.3). The same introduction notes a separate sequel on chains of large gaps; the cited theorem itself supplies no residue-sensitive block estimate of the kind needed here.

What remains to be formalised. Unconditional convergence, the infinite series identity, the actual rational scaled-tail state and its recurrence, the eventual-integral-shift theorem, the abstract rationality classification, and the prime-specific local small-mismatch theorem are Lean-checked. Paper-level are the modular rewriting by $D_{h,N,L}$, the finite-approximation inference, the positive carry countermodel above, the identification of the concrete tail $\widetilde{J}$ with the checked recurrence, and the real form of the small-shift obstruction in Corollary 3.9. The missing statements are the multiple-in-shift lemma used by Problem 5.1, a prime-gap tail domination sharp enough for a certificate, and a cofinal anti-concentration or adjacent-mismatch theorem. No theorem supplies the cofinal pairs in (5.2), and no cited prime-distribution estimate is claimed or formalised as supplying those statements.

Statements and declarations

Artefact and data availability. The pinned formal-source revision contains the Lean sources, the fixed toolchain, and the library manifest used in the verification. This manuscript provides navigation rather than proof authority.

Declaration of generative AI use. Large-language-model agents were used throughout development to draft and revise prose, formal proofs, and software. The author set the objectives and acceptance criteria, selected and reviewed the claims, and approved the published version. The author assumes responsibility for the accuracy, interpretation, and presentation of the work. Generative systems are production tools; they are not authors and supply no independent authority. Lean checks each proof term against the fixed library version, and the sources linked here contain no proof placeholders and no project-defined axioms; Lean does not authorise the exposition, the citation choices, or the interpretation, for which the author remains responsible.

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A Guide to the formal sources

Each linked phrase opens its Lean declaration at the pinned source revision c8e41c76b4ce. All declarations of this note live in one module. The summation-by-parts declarations are prime-specific; most of Section 3 is stated for arbitrary integer digits and arbitrary rational or real orbits, while Section 3.1 records the actual-gap specialisation. The concrete prime-gap tail, its unconditional convergence, its rational candidate state and the real-to-rational scaled-tail bridge are all defined and checked in the pinned Lean module.

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