Excluding the Bounded Negative Part

Lean-checked rigidity for Erdős Problem #243

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July 2026

Abstract

Sylvester's sequence satisfies $a_{n+1} = a_n^2 - a_n + 1$ and has reciprocal sum 1. Erdős asked whether every increasing integer sequence with $a_{n+1}/a_n^2 \to 1$ and rational reciprocal sum satisfies this recurrence eventually. We prove Lean-checked rigidity results for the associated centred denominator-clearing error E_n. Problem #243 remains open: our results do not classify mixed-sign errors or exclude the remaining unbounded-negative regime.

We formalise in Lean three increasingly general exclusions of a negative E. It cannot be constantly $-m$, at any magnitude and any scale: every multiplier would have to share a prime with the fixed m, and once a prime divides one multiplier every later multiplier is 1 modulo it, so a fixed integer would need infinitely many distinct prime divisors. It cannot be periodic in the regime $|E_n| < a_n$ with positive drift, by the same pigeonhole after a strong induction that removes common scale. And, under eventual strict centring and normalised vanishing, it cannot have a bounded negative part: a bounded negative part makes C rise by a bounded amount, normalised vanishing makes C tend to infinity, the tail gcd stabilises so the orbit may be taken reduced, and in a reduced tail the multipliers are pairwise coprime and coprime to every later numerator, which a Chinese-remainder block of consecutive multiples forbids. That last step is a statement about integer sequences alone: a sequence tending to infinity with upward steps bounded by B cannot remain coprime to a fresh pairwise-coprime modulus at every stage. Together with an absorption lemma making $E_n = 0$ propagate, this yields: under exact dynamics, eventual strict centring, an eventually bounded negative part, and normalised vanishing, E vanishes eventually, and the sequence satisfies the Sylvester recurrence eventually. The stated eventual strict-centring hypothesis is redundant, since normalised vanishing yields it after a finite shift. Under the same normalised-vanishing hypothesis, finite normalised negative mass is also impossible.

There is also a weaker conclusion from normalised vanishing alone. It makes strict increases of $\gcd(C_n, D_n)$ sublinear in number and forces constant-gcd blocks of every prescribed finite length arbitrarily far out. It does not make the gcd eventually constant; that stronger conclusion uses cofinally bounded negative magnitudes.

Problem #243 is open and this note does not close it. The last theorem is conditional: its centring and normalised-vanishing hypotheses are analytic inputs that we assume rather than derive. For the canonical orbit attached to a sequence in Erdős's problem, Koizumi's results provide normalised vanishing and a stronger centring range, so the bounded-negative theorem applies conditionally, with boundedness of the negative part as the

Authorship and AI use. The prose, formal proofs, and software in this version were drafted and revised by large-language-model agents under Will Cook's direction. Cook set the objectives and acceptance criteria, reviewed and authorised the manuscript, and is responsible for its contents. The agents are production tools, not authors. See *Statements and declarations*.

missing hypothesis. What the results remove is the bounded case and, more generally, the finite normalised negative-mass case. Under normalised vanishing, summability of $\sum_n (-E_n)_+/C_n$ makes C_n bounded and hence E_n eventually zero. Any surviving canonical orbit must therefore have negative excursions unbounded along a cofinal set and divergent normalised negative mass.

1 Introduction

Sylvester's sequence $2, 3, 7, 43, 1807, \ldots$ is generated by $a_n = a_{n-1}^2 - a_{n-1} + 1$ and has reciprocal sum 1; the recurrence is exactly the statement that the tail beyond a_{n-1} equals $1/(a_{n-1} - 1)$. Erdős Problem #243 asserts that this is the only way a sequence of that growth can have a rational reciprocal sum:

Problem 1.1 (Erdős #243)**.** Let $1 \le a_1 < a_2 < \cdots$ be a sequence of integers with

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}^2} = 1 \qquad \text{and} \qquad \sum \frac{1}{a_n} \in \mathbb{Q}.$$

Then $a_n = a_{n-1}^2 - a_{n-1} + 1$ for all sufficiently large n.

See [3, p. 64] and [4, p. 105]. Bloom's current catalogue record reproduces the displayed problem and labels it open, while explicitly warning that the status is the website owner's present assessment and may not reflect all relevant literature [7]. We therefore use the catalogue for numbering and current reported status only; the original publications carry the mathematical claims. Below we index the same recurrence as $a_{n+1} = a_n^2 - a_n + 1$, which is the shift the formal sources use; the two forms are the same statement. Write $\mathrm{syl}(a) = a^2 - a + 1$, the Sylvester successor.

Relation to prior work. Two published results reach the original problem under analytic hypotheses that are not assumed below. Their hypotheses are conditions on the sequence (a_n) itself. Duverney's is not comparable with the state-level hypotheses used here; the Erdős–Straus one, read through the dictionary of the next paragraph, is implied by nonnegativity of E, the hypothesis of Theorem 4.1, so that criterion already covers the descent proved below (see [6, Cor. 20(1) and Prop. 19(1), pp. 12–13]). Erdős and Straus proved that if $\lim a_n/a_{n-1}^2 = 1$, the reciprocal sum is rational, and a_n does not satisfy the recurrence, then

$$\limsup_{n\to\infty} \frac{[a_1, \ldots, a_n]}{a_{n+1}} \left(\frac{a_{n+1}^2}{a_{n+2}} - 1 \right) > 0,$$

where $[a_1, \ldots, a_n]$ is the least common multiple [1, Theorem 3, p. 132]. This is the indexing in the original theorem. Erdős's 1988 survey, followed by the current catalogue summary, instead prints $a_n^2/a_{n+1} - 1$ in the second factor while retaining the same prefix-LCM quotient; that displayed summary is off by one and is not followed here [4, p. 105][7]. Koizumi records the convenient sufficient rate $a_n^2/a_{n+1} = 1 + o(1/n)$ under which the Erdős–Straus criterion settles the problem [6, Remark 21, p. 13]. Duverney proved a conditional signed form of the problem itself: if $\sum_{n\ge 0} |a_{n+1}/a_n^2 - 1|$ converges,

then the reciprocal sum, even with numerators in $\{-1, 1\}$, is rational if and only if the corresponding signed recurrence holds for all large n [2, Corollary 3.2, p. 287]. The all-positive specialisation is the form relevant here. Neither result is formalised here, and the results below do not recover either of them.

Koizumi's work on doubly exponential sequences [6] bears directly on the state system used below, and supplies the coordinates in which several of the statements of this note are most naturally read. For a rational $r = p/q$ with pseudo-greedy expansion (a_n), remainders (x_n) and gap sequence $\varepsilon_n = x_n^{-1} + 1 - a_n$, his Lemma 15 [6, p. 9] produces integers $c_n > 0$, d_n and e_n with $x_n = c_n/d_n$ and $\varepsilon_n = e_n/c_n$, satisfying

$$a_n = \frac{d_n - e_n}{c_n} + 1, \qquad c_{n+1} = c_n - e_n, \qquad d_{n+1} = a_n d_n,$$

and the proof of that lemma also gives $c_{n+1} = a_n c_n - d_n$. Under the dictionary $(C_n, D_n, E_n) = (c_n, d_n, e_n)$ these are exactly the objects of Section 2: the first relation is the centring map ctr rewritten as $e_n = d_n - (a_n - 1)c_n$, the second is Proposition 2.1, the third is the denominator update, and the fourth is the tail update. The lower-case e_n of Section 2 is $-E_n$, so it is the negative of Koizumi's e_n at an index where the centred state is negative. His Theorem 16 [6, p. 11] shows that his Conjecture 6 — that for a positive rational r whose gap sequence satisfies $\varepsilon_n \to 0$ one has $\varepsilon_n = 0$ for all large n — is equivalent to an affirmative answer to the question of Erdős and Graham recorded as his Question 5, which is Problem 1.1 above. Two of the implications proved below have prior art in these canonical coordinates: his Lemma 13, that $\varepsilon_n = 0$ forces $\varepsilon_{n+1} = 0$, is the absorption of Theorem 3.5, and his Proposition 19(2), that $\varepsilon_n \geq 0$ for all large n forces $\varepsilon_n = 0$ for all large n, is the descent of Theorem 4.1. These appear in canonical coordinates in [6, Lemma 13, p. 9; Prop. 19(2), p. 12]; no question of priority or independence is adjudicated here.

The growth hypothesis in Problem 1.1 is calibrated by two facts about Sylvester's sequence. It satisfies $a_n \approx c_0^{2^n}$ with $c_0 = 1.2640847\ldots$, and shifting it further produces sequences with $a_n \approx C^{2^n}$ for arbitrarily large C whose reciprocals still sum to a rational number [5, arXiv v4, p. 2]. The classical sufficient condition for irrationality, $\lim_n a_n^{1/2^n} = \infty$, is therefore sharp. Kovač and Tao identify that condition as folklore [5, arXiv v4, p. 2]; they attribute the sharpness observation to Erdős (1975). A sequence with $a_n/a_{n-1}^2 \to 1$ has $a_n^{1/2^n}$ convergent, so that criterion says nothing about the sequences of Problem 1.1, and rationality is genuinely possible there. Sylvester's sequence is `A000058` in the OEIS and its shifts `A129871`. For the recent literature on irrationality of Ahmes series we refer to Kovač and Tao [5], who resolve several problems of Erdős and Graham drawn from the same two sources cited above, and whose introduction gives a sample of the intermediate work, including Sándor (1984) and Badea (1987). They do not treat Problem #243; the rigidity conclusion asked for there is not among their results.

The *Formal Conjectures* collection contains a mathematically equivalent unproved declaration, up to its zero-based indexing [8]. Its summand is $\mathbb{Q}$-valued, so Lean's `Summable` hypothesis asserts the existence of a sum in $\mathbb{Q}$; the finite indexing shift changes that sum only by a rational prefix. The declaration therefore does encode the rationality premise, but its proof is `sorry`. We use it as statement-level prior art, not as proof au-

thority; the development below is independent of it. What is machine-checked below concerns the state system rather than the analytic problem: the exclusion of constant and periodic negative magnitudes, the bounded-rise barrier of Theorem 7.3, and the two conditional endpoints (Theorems 8.1 and 9.1), each stated with the exact hypotheses that separate it from Problem #243. No claim of priority is made for anything below.

The hypothesis is asymptotic and the conclusion is exact, so the argument must convert an analytic rate into an integer obstruction. The conversion used here is the classical one: clear denominators along the sequence, so that rationality makes a tail integral, and then centre that integer at the value it would take on Sylvester's sequence. What remains is a single integer error E, and the whole question is whether E can avoid vanishing.

Status. The problem treated here is open, and this note does not close it. Every statement below marked as checked is a proposition that the pinned Lean kernel accepts from the sources this note links to, with no `sorry`, no added axiom, and no unchecked evaluation. That is a claim about the formal statement, not about its mathematical interest, its novelty, or the original problem. The unresolved obligations are named exactly, in their own section, and none of the finite computations, reductions, or no-go results here removes one of them.

Taken together the exclusions constrain the shape of any counterexample. By Theorem 3.5 a counterexample has $E_n \neq 0$ for every large n. By Theorem 4.1 it cannot have E eventually nonnegative. If its tail is all negative, Theorems 5.2 and 6.1 exclude, respectively, constant magnitude and periodic magnitude in the stated regime $e_n < a_n$ with positive drift; they are not a classification of a mixed-sign tail. Under normalised vanishing, Theorems 8.1 and 9.1 show that the negative part is neither bounded nor of finite normalised mass. What survives is stated in Section 10.

Structure

Section 2 sets up the state system, and Section 3 proves the defect identity, from which eventual vanishing of E forces the Sylvester recurrence and a single vanishing error propagates. Section 4 disposes of the case $E \geq 0$ in three lines. The rest of the note is the case $E < 0$, in four widening steps: constant magnitude (Section 5), periodic magnitude (Section 6), bounded magnitude via a coprimality barrier (Sections 7 and 8), finite normalised negative mass (Section 9), and what is left (Section 10). These sections contain the new content; a reader who wants only the strongest statements should read Sections 8 and 9, and then Section 10. Appendix A is a guide to the formal sources, and Appendix B records an exact residue reduction that an earlier finite search used. Linked phrases open the corresponding Lean declaration at the pinned source revision 64f33f3a134d.

Keywords. irrationality; Ahmes series; Sylvester's sequence; unit fractions; Lean 4.
MSC 2020. 11J72 (primary); 11B37, 11D68, 68V20 (secondary).

2 The centred integer state

The three maps defined next carry out the two steps announced at the end of Section 1. The first two clear denominators along the sequence, so that a rational reciprocal sum

leaves a sequence of integers; the third subtracts from that integer the value it takes on Sylvester's sequence, leaving one number whose vanishing is the whole question. All three are maps between integer tuples and carry no hypothesis of any kind.

For $a, D, C \in \mathbb{Z}$ define

$$\mathrm{den}(a, D) = aD, \quad \mathrm{tail}(a, D, C) = aC - D, \quad \mathrm{ctr}(a, D, C) = D - (a-1)C,$$

the denominator update, the tail update, and the centred state. Given a sequence (a_n) of integers, whose terms we call the *multipliers*, we write $D_{n+1} = \mathrm{den}(a_n, D_n)$ for the *denominator state*, $C_{n+1} = \mathrm{tail}(a_n, D_n, C_n)$ for the *tail state*, and $E_n = \mathrm{ctr}(a_n, D_n, C_n)$ for the *centred state*, also called the *error*. The word *centred* records what the third map does: it measures D_n against $(a_n - 1)C_n$, and $D_n = (a_n - 1)C_n$ holds at every index on Sylvester's sequence (Example 2.2), so E is a deviation from that sequence rather than a size. At an index where $E_n < 0$ we write $e_n = -E_n > 0$ and call it the *magnitude* of the error there.

Proposition 2.1 (update law)**.** $\mathrm{tail}(a, D, C) = C - \mathrm{ctr}(a, D, C)$*; that is,* $C_{n+1} = C_n - E_n$*.*

Proof. $C - (D - (a-1)C) = aC - D$. □

Formalised as the update law. The state therefore moves by exactly the error at each step, which is what makes the sign of E decisive: where E is positive the state falls, where E is negative it rises, and where E vanishes it is stationary.

Example 2.2 (the Sylvester orbit)**.** Take $a_n = 2, 3, 7, 43, 1807, \ldots$ with $a_{n+1} = a_n^2 - a_n + 1$, and start the state at $D_0 = C_0 = 1$. The two updates give

n	a_n	D_n	C_n	E_n
0	2	1	1	0
1	3	2	1	0
2	7	6	1	0
3	43	42	1	0
4	1807	1806	1	0

and $D_n = a_n - 1$ with $C_n = 1$ at every index: if $D_n = a_n - 1$ and $C_n = 1$ then $C_{n+1} = a_n - (a_n - 1) = 1$ and $D_{n+1} = a_n(a_n - 1) = a_{n+1} - 1$. Hence $E_n = D_n - (a_n - 1)C_n = 0$ throughout and the tail state never moves, which is Proposition 2.1 in the stationary case.

The intended reading, which is not formalised. Let $T_n = \sum_{k \geq n} 1/a_k$ and suppose $T_0 \in \mathbb{Q}$. Put D_0 equal to a common denominator and $D_n = D_0 a_0 \cdots a_{n-1}$, and set $C_n = D_n T_n$. Then $C_n \in \mathbb{Z}$ for every n, and from $T_n = 1/a_n + T_{n+1}$ one gets $C_{n+1} = a_n C_n - D_n$, which is the tail update. On Sylvester's sequence $T_n = 1/(a_n - 1)$ exactly, so $D_n = (a_n - 1)C_n$ and $E_n = 0$: the centred state measures deviation from the Sylvester tail identity, and it vanishes identically on the Sylvester orbit.

None of this paragraph is formalised. The Lean module contains no series, no rationality hypothesis, and no growth hypothesis; it proves identities and implications

about the integer system above, and about its natural-number realisation, for which the bridging identities are the tail realisation, the denominator realisation, and the signed update law. A reader should treat the identification with reciprocal tails as motivation for the definitions and not as a checked step.

Proposition 2.3 (scale equivariance)**.** *For every $s, a, D, C \in \mathbb{Z}$,*

$$\operatorname{den}(a, sD) = s \operatorname{den}(a, D), \qquad \operatorname{tail}(a, sD, sC) = s \operatorname{tail}(a, D, C),$$

$$\operatorname{ctr}(a, sD, sC) = s \operatorname{ctr}(a, D, C).$$

Formalised as the denominator equivariance, the tail equivariance, and the centred equivariance. The system depends only on the ray through (D, C). This is used twice below: once to normalise the finite search of Appendix B, and once, in Theorem 6.1, as the induction step that removes common scale.

3 The defect identity and its two consequences

Section 2 connects the Sylvester recurrence to the error in one direction only: on Sylvester's sequence the error vanishes. The converse is what the problem needs, and it comes from a single identity, which expresses the deviation of a_{n+1} from the Sylvester successor, multiplied by the tail state C_{n+1}, as a combination of the errors at n and at $n+1$, the first weighted by a_n^2. Its two consequences are Theorem 3.4, in which an eventually vanishing error forces the recurrence provided the tail state is eventually nonzero, and Theorem 3.5, in which, under strict centring (the condition $|E_n| < C_n$ at every index), one vanishing error propagates to every later index.

Write $\operatorname{def}(a, a') = a' - \operatorname{syl}(a)$ for the deviation of the next term from the Sylvester successor, the Sylvester defect.

Theorem 3.1 (defect identity)**.** *For all $a, a', D, C \in \mathbb{Z}$,*

$$\operatorname{def}(a, a') \cdot \operatorname{tail}(a, D, C) = a^2 \operatorname{ctr}(a, D, C) - \operatorname{ctr}(a', \operatorname{den}(a, D), \operatorname{tail}(a, D, C)),$$

that is, $\Delta_n C_{n+1} = a_n^2 E_n - E_{n+1}$.

Proof. A direct expansion: both sides equal $a'aC - a'D - a^3C + a^2D + a^2C - aD - aC + D$. □

Formalised as the defect identity. It is a polynomial identity in four indeterminates, with no hypothesis of any kind, and it carries the whole argument twice over: once for rigidity, and once for absorption.

Example 3.2 (one defect and the error it creates)**.** Continue Example 2.2 but replace $a_3 = 43$ by $a_3 = 44$. The states $D_3 = 42$ and $C_3 = 1$ are unchanged, since they depend only on a_0, a_1, a_2, and $E_2 = 0$ still. The defect is $\Delta_2 = a_3 - \operatorname{syl}(a_2) = 44 - 43 = 1$, and $E_3 = D_3 - (a_3 - 1)C_3 = 42 - 43 = -1$, so the identity reads

$$\Delta_2 C_3 = 1 \cdot 1 = 1 = 49 \cdot 0 - (-1) = a_2^2 E_2 - E_3.$$

By Proposition 2.1 the tail state then rises, $C_4 = C_3 - E_3 = 2$. A single unit of defect at one index has produced a negative error of magnitude 1 at the next, and the state has started to climb.

Theorem 3.3 (two vanishing errors force the step)**.** *Let* $a, a', D, C \in \mathbb{Z}$ *with* $\mathrm{tail}(a, D, C) \neq 0$*. If* $\mathrm{ctr}(a, D, C) = 0$ *and* $\mathrm{ctr}(a', \mathrm{den}(a, D), \mathrm{tail}(a, D, C)) = 0$*, then* $a' = \mathrm{syl}(a)$*.*

Proof. By Theorem 3.1 the product $\mathrm{def}(a, a') \cdot \mathrm{tail}(a, D, C)$ is zero, and the second factor is nonzero by hypothesis. □

Formalised as the local rigidity step. The hypothesis $C_{n+1} \neq 0$ is not removable: at $C_{n+1} = 0$ the identity gives no information about a'.

Theorem 3.4 (sequence form)**.** *Let* $a, D, C : \mathbb{N} \to \mathbb{Z}$ *satisfy* $D_{n+1} = \mathrm{den}(a_n, D_n)$ *and* $C_{n+1} = \mathrm{tail}(a_n, D_n, C_n)$*. If* $E_n = 0$ *for all sufficiently large* n *and* $C_{n+1} \neq 0$ *for all sufficiently large* n*, then* $a_{n+1} = a_n^2 - a_n + 1$ *for all sufficiently large* n*.*

Formalised as the eventual Sylvester recurrence. The proof is routine: take the larger of the two thresholds and apply Theorem 3.3 at each later index. Every later section is directed at the hypothesis of this theorem, that is, at forcing E to vanish eventually.

The defect identity has a second consequence, which is what makes a single vanishing error worth having.

Theorem 3.5 (zero is absorbing)**.** *Let* $a, C, D : \mathbb{N} \to \mathbb{N}$ *be an exact natural orbit, so* $C_{n+1} + D_n = a_n C_n$ *and* $D_{n+1} = a_n D_n$*, and let* $E_n = \mathrm{ctr}(a_n, D_n, C_n)$*. Suppose the centring is strict,* $|E_n| < C_n$ *for every* n*. If* $E_n = 0$ *then* $E_{n+1} = 0$*.*

Proof. Putting $E_n = 0$ in Theorem 3.1 gives $\Delta_n C_{n+1} = -E_{n+1}$, so C_{n+1} divides E_{n+1}. A multiple of C_{n+1} of absolute value smaller than C_{n+1} is zero, and $|E_{n+1}| < C_{n+1}$ by hypothesis. □

Formalised as the absorption of a vanishing centred state, with the companion contrapositive along a tail: if zero is absorbing beyond some index and E does not vanish eventually, then E is nowhere zero beyond that index.

Absorption changes the shape of the problem. Without it, one would have to exclude an E that dips to zero and recovers. With it, either E hits zero once and is finished, or it is never zero at all. Sections 5 and 6 then restrict to the special all-negative case; this is not a without-loss reduction for a mixed-sign tail. The mixed-sign analysis in Sections 8 and 9 instead separates cofinally negative errors from an eventually positive tail.

4 Descent when the error is nonnegative

The hypothesis of the theorem below is the update law of Proposition 2.1 read inside $\mathbb{N}$: the error is nonnegative at every index, so the tail state never rises. This is the easy case, and it is settled in three lines.

Theorem 4.1 (descent)**.** *Let* $C, E : \mathbb{N} \to \mathbb{N}$ *satisfy* $C_{n+1} + E_n = C_n$ *for every* n*. Then* $E_n = 0$ *for all sufficiently large* n*.*

Proof. A one-line descent. The relation forces $C_{n+1} \leq C_n$, so C is a nonincreasing sequence of natural numbers and is eventually equal to its minimum; beyond that index $C_{n+1} = C_n$, and the relation gives $E_n = 0$. □

Formalised as the stabilisation of the nonnegative error, on the eventual constancy of a nonincreasing sequence.

The hypothesis is exactly one-sidedness: C and E take values in $\mathbb{N}$. If $E_n < 0$ at infinitely many indices then C rises there, nothing descends, and the argument is unavailable. That is the whole difficulty, and the rest of this note is directed at it.

5 Excluding a constant negative magnitude

Suppose the error is negative with a constant magnitude, $E_n = -m$ for a fixed $m > 0$ and every n. By Proposition 2.1 the state then climbs by exactly m each step, so $C_n = c + nm$ with $c = C_0$, and the definition of the centred state becomes a *shape equation*

$$D_n + m = (a_n - 1)\,(c + nm). \tag{5.1}$$

Together with $D_{n+1} = a_n D_n$ this is a closed system in (a, D), and it has no solutions. The following instance shows how the failure happens; the theorem then shows that it cannot be avoided.

Example 5.1 (the shape equation running until it fails)**.** Take $m = c = 1$, so that $C_n = 1 + n$ and the shape equation (5.1) reads $D_n + 1 = (a_n - 1)(1 + n)$, and take $D_0 = 1$. Each step is now determined: at $n = 0$ the equation reads $2 = (a_0 - 1) \cdot 1$, so $a_0 = 3$, and $D_1 = a_0 D_0 = 3$; at $n = 1$ it reads $4 = (a_1 - 1) \cdot 2$, so $a_1 = 3$, and $D_2 = a_1 D_1 = 9$; at $n = 2$ it reads $10 = (a_2 - 1) \cdot 3$, which has no integer solution, and the orbit stops. Longer prefixes occur for other starting values: the finite search recorded in Appendix B found none longer than 17 among the initial states below 5000.

Theorem 5.2 (no constant negative magnitude)**.** *There is no pair of sequences $a, D : \mathbb{N} \to \mathbb{N}$ with $a_n \geq 2$ for all n satisfying $D_{n+1} = a_n D_n$ and (5.1), for any $m > 0$ and any c. The same holds if the shape only begins at some index.*

Proof when c and m are coprime. Assume first $\gcd(c, m) = 1$. Two facts about the multipliers do all the work, and the difficulty lies in the fact that they concern the same finite set of primes from opposite directions: the first makes every multiplier meet that set, the second lets each prime of the set meet at most one multiplier.

Every a_j shares a prime with m. Suppose $\gcd(a_j, m) = 1$. Then m is invertible modulo a_j, so some n has $c + nm \equiv 0$, that is $a_j \mid C_n$; and $a_j \mid D_n$ for every $n > j$, since D is multiplicative with a_j among its factors. Choosing such an n beyond j and reading (5.1) modulo a_j gives $a_j \mid m$, contradicting $\gcd(a_j, m) = 1$ and $a_j \geq 2$.

A prime divisor of m occurs in at most one a_j. Suppose $p \mid m$ and $p \mid a_j$. For $n > j$ we have $p \mid D_n$, so (5.1) gives $(a_n - 1)C_n \equiv 0 \pmod{p}$. Now $C_n = c + nm \equiv c$, and $p \nmid c$ because $p \mid m$ and $\gcd(c, m) = 1$; hence $p \mid a_n - 1$, so $p \nmid a_n$. Thus p cannot divide any later multiplier.

The two facts are incompatible. Each of the infinitely many multipliers needs a prime divisor of m, and each prime divisor of m serves at most one multiplier, while m has only finitely many prime divisors. □

The case $\gcd(c, m) = 1$ is the scale-primitive exclusion. The special case $m = c = 1$, worked through in Example 5.1, where m has no prime divisors at all and the first fact is immediately contradictory, is recorded separately as the normalised exclusion.

Removing the scale. For general c, put $g = \gcd(c, m)$. Equation (5.1) shows $g \mid D_n$ for every n, so dividing D, c and m by g leaves a system of the same shape with coprime data, which the previous case excludes. □

Formalised as the exclusion at every scale, and the eventual form, obtained by shifting the orbit to the first constant index, as the eventual exclusion.

Theorem 5.2 excludes a centred state that is eventually constant and negative, at every magnitude and every scale. It says nothing about a negative state whose magnitude changes, and the argument genuinely uses constancy: the finiteness of the set of prime divisors of m is what the pigeonhole uses, and a varying magnitude presents a fresh set at each index.

6 Excluding a periodic negative magnitude

The next case allows the magnitude to vary, provided it repeats. Write $e_n = -E_n > 0$ for the magnitude, as in Section 2, so that $C_{n+1} = C_n + e_n$; suppose e has period h, meaning $e_{n+h} = e_n$ for every n, and that the state gains a fixed *drift* $M > 0$ over one period, meaning $C_{n+h} = C_n + M$ for every n.

At $h = 1$ this says that the magnitude is constant and that M is its value, which is the situation of Section 5; Theorem 5.2 already excludes it, and does so without the extra hypothesis $e_n < a_n$ imposed below. The new content is $h \geq 2$.

Section 5 could run its pigeonhole on the prime divisors of the single number m. Here the magnitude changes inside a period, and the number available to run it on is the drift M. Three lemmas replace the two facts of Section 5. The first is that multipliers persist: every a_j divides every strictly later denominator state, the persistence of a multiplier. The second is a *lock*: a prime already dividing D and also dividing the current multiplier is forced into the next tail state, the one-step lock, and once present in both D and C it stays in every later D, C and magnitude, the persistence of a lock. The third transports a lock backwards through the period: if a divisor of the drift M occurs in two multipliers, it is a common divisor of C_0 and of every magnitude, the repeated-divisor transport, using that both the period relation and the drift iterate over any number of periods, the iterated period and the iterated drift.

Theorem 6.1 (no periodic negative magnitude)**.** *Let $a, D, C, e : \mathbb{N} \to \mathbb{N}$ with $a_n \geq 2$, $e_n > 0$ and $e_n < a_n$ for every n, satisfying*

$$D_{n+1} = a_n D_n, \qquad C_{n+1} = C_n + e_n, \qquad D_n + e_n = (a_n - 1)C_n,$$

and suppose $e_{n+h} = e_n$ and $C_{n+h} = C_n + M$ for some $h > 0$ and $M > 0$. This is impossible.

Proof. Strong induction on the drift M. The difficulty lies in the fact that a repeated prime divisor of M need not contradict anything directly. It instead forces a common

scale on the whole orbit, which can be divided out, and the induction runs on the drift that the division reduces.

Suppose first that no prime divisor of M is a common divisor of C_0 and of every magnitude. Repeated-divisor transport then applies in the contrapositive: a prime divisor of M occurring in two of the multipliers would be exactly such a common divisor, so each prime divisor of M occurs in at most one multiplier. Against this, the argument of Section 5 run phase by phase (along a fixed residue class modulo h the magnitude is constant and C advances by M at each period, so the multipliers of that phase meet an arithmetic progression exactly as in Section 5) makes every multiplier share a prime with M. Since M has only finitely many prime divisors and there are infinitely many multipliers, pigeonhole gives a contradiction.

Otherwise some prime p divides M, divides C_0, and divides every magnitude. Then $p \mid C_n$ for every n, since $C_{n+1} = C_n + e_n$ and both summands on the right are divisible by p, and the shape equation $D_n + e_n = (a_n - 1)C_n$ then gives $p \mid D_n$ for every n. Divide D, C and e by p. The multipliers are untouched, so $a_n \geq 2$ and $e_n < a_n$ persist and the magnitudes stay positive; the three recurrences are homogeneous in (D, C, e) and so survive; the period is still h; and the drift becomes $M/p < M$. The inductive hypothesis applies. □

The first of the two cases above is the phase-primitive exclusion, the induction is the exclusion at every scale, and the eventual form is the eventual exclusion.

The bound $e_n < a_n$ is a genuine restriction and not a normalisation. It is the range in which a multiplier cannot divide its own phase magnitude, which is what the lock argument needs. It holds in the intended reading, where e_n is a centred representative and a_n grows quadratically, but it is assumed here and not derived. The drift hypothesis $M > 0$ is also used: a periodic magnitude with zero drift would be identically zero, which is the case already settled by Section 4.

Remark. On the canonical orbit of [6], read through the dictionary of Section 1, the bound $e_n < a_n$ holds eventually. Lemma 15(2) there gives the centring range $-C_n/2 \leq E_n < C_n/2$, so $e_n = |E_n| \leq C_n/2$ at a negative index, and the proof of Lemma 15(3) gives $C_{n+1} \leq \frac{3}{2}C_n$ [6, p. 9], hence the uniform bound $C_n \leq C_0(3/2)^n$. Against this, summability of $\sum 1/a_n$ forces $a_n \to \infty$, and the growth hypothesis $a_n^2/a_{n+1} \to 1$ gives $a_{n+1} \geq a_n^2/2$ for all large n, hence $a_{n+1} \geq 2a_n$ once $a_n \geq 4$; so the multipliers grow at least geometrically with ratio 2 and outgrow the bound on C_n, making e_n/a_n eventually small. Two caveats. Summability is used for this step, so the estimate is not available from the state recurrence alone; and the conclusion is eventual and not universal, whereas Theorem 6.1 imposes $e_n < a_n$ at every index, so it is the eventual form of the exclusion cited above that applies. Under the periodicity hypothesis of Theorem 6.1 the estimate is in any case unnecessary, since a periodic magnitude is bounded while the multipliers tend to infinity.

7 Bounded rise cannot remain coprime to fresh moduli

Periodicity is still a strong assumption. Removing it needs a different kind of obstruction, and the one used here is a counting argument about where an integer sequence with bounded upward steps must land.

Lemma 7.1 (shifted blocks of consecutive multiples)**.** *Let $m_0, \dots, m_{B-1}$ be pairwise coprime and at least 2. For every bound there is a t beyond it with $m_i \mid t + i$ for each $i < B$.*

Proof. Standard Chinese remaindering: solve $t \equiv -i \pmod{m_i}$ simultaneously, then add multiples of $\prod_i m_i$ to pass the bound. □

Formalised as the shifted consecutive multiples.

Example 7.2 (a block of three consecutive multiples)**.** Take $B = 3$ and $(m_0, m_1, m_2) = (3, 4, 5)$. The congruences $t \equiv 0 \pmod 3$, $t \equiv 3 \pmod 4$ and $t \equiv 3 \pmod 5$ hold exactly when $t \equiv 3 \pmod{60}$. At $t = 3$ the block is $3, 4, 5$ with $3 \mid 3$, $4 \mid 4$ and $5 \mid 5$; at $t = 63$ it is $63, 64, 65$ with $3 \mid 63$, $4 \mid 64$ and $5 \mid 65$. A sequence of natural numbers that starts at 0, tends to infinity and rises by at most 3 at each step cannot step over the block $\{3, 4, 5\}$: it has a first value at least 3, that value is at most 5, and every member of $\{3, 4, 5\}$ is divisible by one of the three moduli.

Theorem 7.3 (bounded rise cannot remain coprime to fresh moduli)**.** *Let $u : \mathbb{N} \to \mathbb{N}$ tend to infinity with $u_{n+1} \le u_n + B$ for a fixed $B > 0$. Then u cannot remain coprime to infinitely many fresh pairwise coprime moduli: there is no family of pairwise coprime $m_i \ge 2$, one for each index, such that $\gcd(m_i, u_t) = 1$ whenever $i < t$.*

Proof. Take B of the moduli, from indices beyond any chosen point, and use Lemma 7.1 to find t far out with the ith of them dividing $t + i$. The interval $[t, t + B)$ has length B, and u climbs by at most B per step while tending to infinity, so some u_s lands in it. Then $u_s = t + i$ for some $i < B$, and the ith modulus divides u_s. Since that modulus is at least 2, this contradicts $\gcd(m_i, u_s) = 1$. □

Formalised as the bounded-rise barrier. The key point is that the two hypotheses pull against each other exactly: divergence forces u to travel arbitrarily far, and bounded rise forbids it from stepping over a window of width B. Uniformity of the bound is what the proof uses, and it is the hypothesis that cannot be relaxed by this argument: a rise bounded only along a subsequence leaves the intervening steps free to jump the window. We have not looked for a sequence of unbounded rise remaining coprime to every fresh modulus. The statement is elementary and we are not aware of a prior published form of it, but we would not be surprised if one exists.

The moduli are indexed by the same set as the sequence, and the ith of them is asked to divide no value u_t with $t > i$; nothing at all is asked of it at or before its own index. That asymmetry is what makes the hypothesis available in the application below, where m_i is the ith multiplier and only the later numerators are known to be coprime to it.

The barrier applies to the problem because a *reduced* exact tail already carries pairwise coprime moduli. Call (u, v) a reduced exact tail with multipliers a when $\gcd(u_n, v_n) = 1$,

$$u_{n+1} + v_n = a_n u_n, \qquad v_{n+1} = a_n v_n.$$

Here u is the numerator and v the denominator: the two recurrences are those of Section 2, with u in the role of the tail state C and v in that of the denominator state D, and with coprimality imposed at every index.

Proposition 7.4 (reduced tails are pairwise coprime)**.** *In a reduced exact tail, a_n is coprime to v_n; the multipliers at distinct indices are pairwise coprime; and every earlier multiplier is coprime to every later numerator.*

These are the step coprimality, the pairwise coprimality, and the whole-modulus avoidance. Feeding them to Theorem 7.3 gives the form used later: there is no reduced exact tail whose numerator tends to infinity with a uniformly bounded upward increment, the reduced bounded-rise exclusion, together with its eventual form, the eventual reduced exclusion.

Example 7.5 (Sylvester's sequence as a reduced exact tail)**.** Put $u_n = 1$ and $v_n = D_n = 1, 2, 6, 42, 1806, \ldots$ as in Example 2.2, with multipliers $a_n = v_n + 1 = 2, 3, 7, 43, 1807, \ldots$. Then $\gcd(u_n, v_n) = 1$, $u_{n+1} + v_n = 1 + v_n = a_n u_n$ and $v_{n+1} = a_n v_n$, so this is a reduced exact tail, and Proposition 7.4 returns the classical fact that Sylvester's numbers are pairwise coprime. Its numerator is constant, so it satisfies every hypothesis of the exclusion just stated except divergence; since the tail exists, that hypothesis cannot be dropped.

An arbitrary orbit of Section 2 need not be reduced, so one more step is needed before the barrier can be applied to it: the common factor must stop changing. We call a set of indices *cofinal* when it is infinite, and say that a property holds *cofinally* when the set of indices at which it holds is cofinal. The step below is the only one connecting the orbit of Section 2 to the reduced tails of Proposition 7.4, and it is where the cofinal boundedness in the proof of Theorem 8.1 is used.

Proposition 7.6 (the tail gcd stabilises)**.** *Let (a, D, C) be an exact natural orbit, that is, a triple of $\mathbb{N}$-valued sequences with $C_{n+1} + D_n = a_n C_n$ and $D_{n+1} = a_n D_n$, whose centred state $E_n = \mathrm{ctr}(a_n, D_n, C_n)$ is negative cofinally, with magnitudes bounded along that cofinal set. Then $\gcd(C_n, D_n)$ is eventually constant, and beyond that index the orbit divided by the stable gcd is a reduced exact tail.*

Proof. The tail gcd $\gcd(C_n, D_n)$ divides its successor, the gcd monotonicity, so the tail gcds form a positive divisibility chain. At an index where the centred state is negative the tail gcd is exactly $\gcd(C_n, e_n)$, the gcd at a negative index, and is therefore at most the magnitude there; so the chain is bounded along the cofinal set of negative indices. A positive divisibility chain whose values are bounded along a cofinal set is eventually constant, the chain stabilisation. Hence cofinally bounded negative magnitudes make the tail gcd stabilise, the gcd stabilisation, and beyond that point dividing by the stable gcd leaves the numerator and denominator coprime, that is, a reduced exact tail. □

The stabilisation of the tail gcd is the checked statement cited at the end of the proof; the last clause, that the divided orbit is a reduced exact tail, is exposition and carries no separate label. The hypothesis is boundedness *along a cofinal set* and not boundedness at every index. That is what the situation gives: the tail gcd is controlled only at the negative indices, and cofinally many of them is all the divisibility chain needs.

Normalised vanishing by itself gives a different, strictly weaker conclusion. For an exact natural orbit with $C_n > 0$, put

$$G_n = \gcd(C_n, D_n), \qquad \Gamma(N) = \#\{0 \le j < N : G_j < G_{j+1}\}.$$

Proposition 7.7 (normalised vanishing makes strict gcd changes sparse)**.** *Assume*

$$(\forall K \ge 1)(\exists N)(\forall n \ge N) \qquad K|E_n| < C_n.$$

Then, for every $K \ge 1$, eventually $K\Gamma(N) < N$. Moreover, for every starting bound B and block length L, some $n \ge B$ satisfies

$$G_n = G_{n+1} = \cdots = G_{n+L}.$$

The first assertion is the sublinear strict-growth theorem; the second is the arbitrarily late constant-block theorem. The proof first converts normalised vanishing into subexponential growth of C_n, then charges every strict divisibility increase of G_n against a finite power-of-two budget. Sparse strict changes force long finite gaps between them.

The quantifiers matter. Proposition 7.6 uses cofinally bounded negative magnitudes and yields eventual constancy. Proposition 7.7 uses only normalised vanishing and yields arbitrarily late constant blocks of each prescribed finite length. It does not yield one infinite constant tail, and it does not exclude cofinally unbounded negative excursions.

8 Excluding a bounded negative part

Assembling the previous section gives Theorem 8.1, the endpoint of the barrier argument. Theorem 9.1 in the next section is a second endpoint under a different finiteness hypothesis, and neither theorem contains the other. Theorem 9.1 assumes no orbit structure, only the state recurrence with $C_n > 0$, so it is not a special case of Theorem 8.1; conversely hypothesis (5) below does not imply finite *normalised negative mass*, by which we mean convergence of $\sum_n (-E_n)_+/C_n$, where $x_+ = \max(x, 0)$, since a constant negative part $-b$ gives $C_n \sim bn$ and hence a divergent sum. The hypotheses of Theorem 8.1 are listed in full, because two of them are analytic inputs that are assumed here rather than derived from the growth condition of Problem 1.1.

Informally, hypotheses (5) and (6) below pull against each other. By Proposition 2.1, hypothesis (5) caps the rise of the tail state at B per step, while hypothesis (6), once the error is nowhere zero, forces the tail state to diverge. Proposition 7.6 makes the orbit reduced, and Theorem 7.3 then converts the tension into a contradiction: an integer sequence tending to infinity with upward steps of size at most B cannot remain coprime to the pairwise coprime moduli a reduced tail carries. The complementary case, in which the error is eventually positive, is the descent of Theorem 4.1.

Theorem 8.1 (bounded negative part)**.** *Let $a, C, D : \mathbb{N} \to \mathbb{N}$ and $E : \mathbb{N} \to \mathbb{Z}$ satisfy*

1. *$a_n > 1$ and $C_n > 0$ for every n;*
2. *the exact dynamics $C_{n+1} + D_n = a_n C_n$ and $D_{n+1} = a_n D_n$;*

3. $E_n = \mathrm{ctr}(a_n, D_n, C_n)$ *for every* n;
4. eventual strict centring*:* $|E_n| < C_n$ *for all large* n;
5. eventually bounded negative part*:* $-B \le E_n$ *for all large* n*, for some* B;
6. normalised vanishing*: for every* K *there is an* N *with* $K|E_n| < C_n$ *for all* $n \ge N$.

Then $E_n = 0$ *for all sufficiently large* n.

Proof. Suppose not. We first remove every late zero: by Theorem 3.5 and its contrapositive form, E is nowhere zero beyond the centring threshold. Two cases remain. If E is negative cofinally, then (5) bounds those magnitudes, so Proposition 7.6 makes the tail gcd stabilise and the orbit may be taken reduced past that point. Hypothesis (5) with Proposition 2.1 gives $C_{n+1} \le C_n + B$, a bounded rise; hypothesis (6) with E nowhere zero gives $C_n \to \infty$. A reduced exact tail with a divergent numerator and bounded rise is excluded by Theorem 7.3. It remains to treat the case that E is eventually positive, and there the descent of Theorem 4.1 applies and forces E to vanish, contradicting nowhere-vanishing. □

Formalised as the bounded-negative-part rigidity, with the eventual form, in which the centring and the bound need only hold from some index, as the eventual bounded-negative-part rigidity. The two auxiliary results it uses are the divergence from normalised vanishing and the exclusion of cofinally bounded negative magnitudes, the latter over the tail-divergence form.

Corollary 8.2. *Under the hypotheses of Theorem 8.1, together with* $C_{n+1} \neq 0$ *for all large* n*, the multipliers satisfy* $a_{n+1} = a_n^2 - a_n + 1$ *for all sufficiently large* n.

Proof. Theorem 8.1 then Theorem 3.4. □

Hypotheses (1)–(3) are the state system, and (6) is the division-free form of $|E_n|/C_n \to 0$. Hypothesis (4) is logically redundant: hypothesis (6) with $K = 1$ gives $|E_n| < C_n$ eventually. It is retained because the displayed statement tracks the linked Lean declaration; shifting to the resulting threshold gives the same conclusion without an independent centring input. The formal source does not derive normalised vanishing from $a_{n+1}/a_n^2 \to 1$, so Theorem 8.1 is, by itself, a conditional theorem about the state system. Combined with Koizumi's bridge below, however, Corollary 8.2 settles the bounded-negative canonical case of Problem 1.1. What is unconditional in this note is the all-negative constant and periodic exclusions, Theorems 5.2 and 6.1, and the barrier itself, Theorem 7.3.

The conditionality can nevertheless be located precisely, because the missing inputs are supplied elsewhere. With the results of [6], Theorem 8.1 becomes a conditional theorem about Problem 1.1 itself rather than about the state system alone. Let (a_n) satisfy the hypotheses of Problem 1.1. After deleting a finite prefix, Corollary 10 of [6, p. 8] makes the sequence the pseudo-greedy expansion of its own reciprocal sum, and Lemma 15 there supplies the integers c_n, d_n, e_n of Section 1. Hypothesis (1) then holds: c_n is a positive integer by Lemma 15(1), and $a_n > 1$ after a further finite shift, because summability of $\sum 1/a_n$ forces $a_n \to \infty$. Hypothesis (2) is the pair of recurrences $c_{n+1} = a_n c_n - d_n$ and

$d_{n+1} = a_n d_n$ of Lemma 15(2), and hypothesis (3) is that lemma's $a_n = (d_n - e_n)/c_n + 1$ read as $e_n = d_n - (a_n - 1)c_n$, which is the map ctr. Hypothesis (4) holds at every index, not merely eventually, from the centring range $-c_n/2 \le e_n < c_n/2$ of the same lemma. Hypothesis (6) is the vanishing of the gap sequence in Corollary 10, since $\varepsilon_n = e_n/c_n$. The sole hypothesis not supplied is (5), the eventual bound on the negative part; nothing in [6] yields it and nothing here does either, so Problem #243 stays open.

9 Excluding finite normalised negative mass

Theorem 8.1 uses a uniform bound on the negative part. A different finiteness hypothesis, on the total normalised negative mass, removes a further family of orbits, and the argument is multiplicative rather than arithmetic: it compares C with a convergent product instead of with a system of congruences.

Theorem 9.1 (finite normalised negative mass)**.** *Let $C_n \in \mathbb{N}_{>0}$ and $E_n \in \mathbb{Z}$ satisfy*

$$C_{n+1} = C_n - E_n, \qquad \forall K\, \exists N\, \forall n \ge N, \quad K|E_n| < C_n.$$

If

$$\sum_{n=0}^{\infty} \frac{(-E_n)_+}{C_n} < \infty,$$

then $E_n = 0$ eventually. For an exact reciprocal-tail orbit this implies $a_{n+1} = a_n^2 - a_n + 1$ eventually.

Proof. Put $\delta_n = (-E_n)_+/C_n$. The update gives $C_{n+1} \le C_n(1 + \delta_n)$, so

$$C_N \le C_0 \prod_{n<N} (1 + \delta_n).$$

The product is bounded because $\sum \delta_n$ converges. Choose an integer M above that bound and apply normalised vanishing with $K = M$. Then $M|E_n| < C_n < M$ for all sufficiently large n, forcing the integer E_n to be zero. The final recurrence is Theorem 3.4, whose hypothesis $C_{n+1} \ne 0$ holds because $C_n > 0$ throughout. □

The product argument is the summable relative growth; its specialisation to negative mass is the vanishing from finite negative mass, and the statement giving the recurrence directly is the Sylvester recurrence from summable negative mass. This is a checked implication about the state system, not a derivation of normalised vanishing from Problem #243. Normalised vanishing is still an input: it is what converts the bound on C into a bound on $|E_n|$, and nothing here derives it from the growth condition of Problem 1.1.

The transfer of the previous section applies here as well. On the canonical orbit of Corollary 10 of [6, p. 8] the recurrence $C_{n+1} = C_n - E_n$ holds with $C_n > 0$ by Lemma 15 [6, p. 9], and normalised vanishing is the vanishing of the gap sequence, so Theorem 9.1 is a conditional theorem about Problem 1.1 whose sole remaining hypothesis is summability of $\sum_n (-E_n)_+/C_n$. That hypothesis is not supplied there, and is the content of Problem 10.5 below.

Example 9.2 (the product bound at a geometric rate)**.** Suppose $C_0 = 100$ and $(-E_n)_+/C_n \le 2^{-n-1}$ for every n, so that the normalised negative mass is at most 1; the constraint at $n = 0$ still permits E_0 as negative as -50. Since $1 + x \le e^x$,

$$C_N \le 100 \prod_{n<N} (1 + 2^{-n-1}) \le 100e < 272$$

for every N. Normalised vanishing at $K = 272$ then gives $272|E_n| < C_n < 272$ for all large n, so $|E_n| < 1$ and $E_n = 0$ there. The constant 272 comes from the total mass and not from any single magnitude, which is why the hypothesis of Theorem 9.1 is summability of $(-E_n)_+/C_n$ and not a bound on it.

10 Complements and further questions

Problem #243 is open. The negative case has narrowed, and it is worth being precise about what is left.

Each exclusion holds only in its stated regime. Theorems 5.2 and 6.1 exclude the constant and the stated periodic magnitudes on an all-negative tail, and do not exclude those patterns merely along the negative indices of a mixed-sign tail; Theorems 8.1 and 9.1 exclude a bounded negative part and finite normalised negative mass under normalised vanishing. Together with absorption and descent they fix the profile of any counterexample.

Proposition 10.1 (frontier profile for a counterexample)**.** *For the canonical integer state attached to any counterexample to Problem 1.1,*

$$E_n \neq 0 \quad \textit{eventually}, \qquad \frac{|E_n|}{C_n} \longrightarrow 0,$$

and

$$\limsup_{\substack{n\to\infty \\ E_n<0}} (-E_n) = \infty, \qquad \sum_{n=0}^{\infty} \frac{(-E_n)_+}{C_n} = \infty.$$

In particular, negative indices occur infinitely often. Thus the remaining regime consists of globally coherent, unbounded negative excursions with divergent normalised negative mass.

Proof. Koizumi's canonical-tail transfer gives strict centring and normalised vanishing after a finite shift; the strict-centring hypothesis of Theorem 8.1 also follows from normalised vanishing with $K = 1$. Absorption then makes $E_n \neq 0$ eventually, since eventual zero would give the Sylvester recurrence. Descent excludes an eventually nonnegative state. Theorem 8.1 excludes a bounded negative part, and Theorem 9.1 excludes finite normalised negative mass. These statements are unchanged by deleting a finite prefix. □

Put

$$\delta_n := \frac{(-E_n)_+}{C_n}.$$

Every argument in Sections 5–8 uses a finiteness hypothesis: a fixed set of prime divisors, a fixed period, or a fixed bound on the negative part. Theorem 9.1 uses $\sum_n \delta_n < \infty$.

Proposition 10.1 records that a counterexample has none of them; it is a frontier statement, not an additional problem equivalent to the original one. Stated at the level of state orbits, the surviving obstruction is the following.

Problem 10.2 (unbounded divergent-mass negative excursions)**.** Exclude, or construct, an exact natural orbit (a, D, C) with $a_n > 1$ and $C_n > 0$, whose multipliers satisfy $\lim_n a_{n+1}/a_n^2 = 1$ and whose centred state satisfies normalised vanishing, and which is negative infinitely often with magnitudes unbounded along that cofinal set and

$$\sum_n \frac{(-E_n)_+}{C_n} = \infty.$$

An excursion of this kind has none of the finiteness hypotheses just listed. By Proposition 10.1 the canonical state of a counterexample is such an orbit after deletion of a finite prefix, so an exclusion would settle Problem #243. A construction would not settle it in the other direction: an exact state orbit need not arise from a sequence satisfying Problem 1.1.

Remark. The hypotheses in Problem 10.2 beyond the state recurrence are not decorative: the recurrence alone admits unbounded divergent-mass excursions. Put

$$C_n = 2^n, \qquad b_0 = 2, \qquad b_{n+1} = \tfrac{1}{2} b_n (b_n + 2), \qquad a_n = b_n + 2, \qquad D_n = b_n C_n,$$

so that $(b_n) = 2, 4, 12, 84, 3612, \ldots$ and $(a_n) = 4, 6, 14, 86, 3614, \ldots$. Every b_n is a positive even integer, since $b_n = 2k$ gives $b_{n+1} = 2k(k+1)$, so all four sequences are $\mathbb{N}$-valued. The orbit is exact: $C_{n+1} + D_n = 2^n(2 + b_n) = a_n C_n$ and $D_{n+1} = b_{n+1} 2^{n+1} = b_n(b_n + 2)2^n = a_n D_n$. Its centred state is

$$E_n = D_n - (a_n - 1)C_n = b_n 2^n - (b_n + 1)2^n = -C_n,$$

so the negative magnitudes 2^n are unbounded and $\sum_n (-E_n)_+ / C_n$ diverges. What fails is the centring: $|E_n| = C_n$ at every index, so strict centring fails at the boundary and normalised vanishing fails outright. The multipliers satisfy $a_{n+1} = \frac{1}{2}(a_n^2 - 2a_n + 4)$, so $a_{n+1}/a_n^2 \to \frac{1}{2}$ and the growth hypothesis fails as well. The example therefore says nothing about Problem 1.1, and it does not isolate the centring hypotheses: $E_n = -C_n$ at every index forces that multiplier recurrence, so growth fails with them. It is recorded only to show that the state recurrence alone is not enough.

10.1 Global height and old-factor overlap

The main arithmetic gap is global. Define the cumulative digit LCM and its overlap quotient by

$$L_0 = D_0, \qquad L_{n+1} = \operatorname{lcm}(L_n, a_n), \qquad M_n = \frac{D_n}{L_n}.$$

Since $D_n = D_0 \prod_{j<n} a_j$, the quotient is integral and $M_n L_n = D_n$. It records the multiplicity lost when the full product scale is compressed to an LCM.

Problem 10.3 (global overlap-height growth)**.** For every nonterminal canonical orbit satisfying Problem 1.1, must

$$\limsup_{n\to\infty} \frac{\log M_n}{n} > 0?$$

Equivalently, must there be a $K \geq 1$ for which

$$2^n \leq M_n^K$$

at infinitely many indices?

The two displayed formulations are equivalent up to changing the positive constant; the second is not a weaker target. A positive answer is not yet, by itself, a proof of Problem #243: the missing bridge is a comparison of M_n with a subexponential state or exact cancellation-payment scale. No assertion such as $M_n \mid C_n$ or $M_n \leq C_n$ is used here. Proving such a bridge together with Problem 10.3 would collide with the subexponential tail-height budget forced by normalised vanishing.

There is also a more local-looking question whose content is nevertheless the entire prefix. Put $A_n = \prod_{j<n} a_j$, so $D_n = D_0 A_n$. From

$$D_0 A_n = (a_n - 1)C_n + E_n$$

one immediately obtains

$$\gcd(A_n, a_n - 1) \mid E_n.$$

The checked tail-height estimate and normalised vanishing make $|E_n|$ subexponential in n.

Problem 10.4 (old-factor overlap)**.** In every nonterminal canonical orbit satisfying Problem 1.1, is

$$\limsup_{n\to\infty} \frac{\log \gcd(A_n, a_n - 1)}{n} > 0?$$

Equivalently, do there exist $\eta > 0$ and infinitely many n such that $\gcd(A_n, a_n - 1) \geq e^{\eta n}$?

A positive answer contradicts the displayed divisibility and the subexponential error budget. Arbitrarily long locally admissible blocks do not answer this question: the gcd uses the complete prefix.

10.2 The direct analytic question

Problem 10.5 (summability from rationality)**.** Let (a_n) satisfy Problem 1.1, and let (D_n, C_n, E_n) be its canonical integer state. Must

$$\sum_{n=0}^{\infty} \delta_n = \sum_{n=0}^{\infty} \frac{(-E_n)_+}{C_n} < \infty?$$

An affirmative answer closes Problem #243 by Theorem 9.1. Since $\delta_n \to 0$, convergence is equivalently expressible as boundedness of the partial products $\prod_{n<N}(1+\delta_n)$; that is an explanatory reformulation, not a second success criterion. A negative answer must be a sequence satisfying the full growth and rationality hypotheses, not merely a locally admissible state orbit.

Remark. One further question is not left open here, since it is answered in [6]: whether normalised vanishing follows from $a_{n+1}/a_n^2 \to 1$ and rationality. It does. After deleting a finite prefix, Koizumi's Corollary 10 [6, p. 8] makes the sequence the pseudo-greedy expansion of its own reciprocal sum and gives $\varepsilon_n \to 0$ under exactly the hypotheses of Problem 1.1; rationality is not needed for that step, only summability of $\sum 1/a_n$. Since $\varepsilon_n = E_n/C_n$ under the intended reading of Section 2, this is normalised vanishing. Strict centring comes from the same source and is stronger than what Theorem 8.1 assumes: for a rational reciprocal sum, the range $-c_n/2 \le e_n < c_n/2$ of his Lemma 4(2) [6, p. 11] gives $|E_n| < C_n$ at every index, not merely for all large n; eventual strict centring also follows directly from normalised vanishing with $K = 1$. What remains unsupplied is hypothesis (5), the bound on the negative part, which is why the theorem is still conditional.

10.3 A barrier extension

Problem 10.6 (variable-rise CRT barrier)**.** Determine the weakest growth condition on the positive increments of u_n under which an unbounded integer sequence cannot remain coprime to infinitely many pairwise-coprime old moduli as in Theorem 7.3. In particular, does the conclusion remain valid under the candidate condition

$$(u_{n+1} - u_n)_+ = o(\log\log(u_n + 3))?$$

The displayed rate is a proposed threshold, not a proved sharp boundary. Either an extension of the CRT barrier or a counterexample with this rise rate would be informative independently of Problem #243.

Formalisation targets

Erdős–Straus. Formalise Theorem 3 of [1], with its prefix-LCM quotient and correctly indexed growth factor, under the published analytic hypotheses.

Duverney. Formalise Corollary 3.2 of [2], including its signed numerators and absolute convergence hypothesis, and then recover the all-positive specialisation used here. These are useful verification projects, but neither is an additional open mathematical problem.

Statements and declarations

Artefact and data availability. The pinned formal-source revision contains the Lean sources, the fixed toolchain, and the library manifest used in the verification. This manuscript provides navigation rather than proof authority.

Declaration of generative AI use. Large-language-model agents were used throughout development to draft and revise prose, formal proofs, and software. The author set the objectives and acceptance criteria, selected and reviewed the claims, and approved the published version. The author assumes responsibility for the accuracy, interpretation, and presentation of the work. Generative systems are production tools; they are not authors and supply no independent authority. Lean checks each proof term against the fixed library version, and the sources linked here contain no proof placeholders and

no project-defined axioms; Lean does not authorise the exposition, the citation choices, or the interpretation, for which the author remains responsible.

Funding and competing interests. This work received no external funding. The author declares no competing interests.

Acknowledgements. The problem numbering and status follow the Erdős Problems catalogue maintained by Thomas Bloom [7].

A Guide to the formal sources

Each linked phrase opens its Lean declaration at the pinned source revision 64f33f3a134d. The state system, the exclusions, the barrier, and the bounded-negative-part theorem are in `ReciprocalTailRigidity.lean`; the finite-negative-mass theorem is in `SparseResetRecovery.lean`; and the residue reduction of Appendix B is in `FiniteHorizonResidue.lean`. Three distinctions are worth carrying into the source. The identification of any of these modules with reciprocal tails is the exposition of Section 2 and is not a checked statement. The periodic exclusion assumes the regime $e_n < a_n$. And the final Lean declaration lists strict centring and normalised vanishing as hypotheses; the former follows eventually from the latter with $K = 1$, while the formal source derives neither from the original analytic problem. For the external bridge supplying them, see Sections 1 and 8.

B A factorial residue reduction for forced orbits

In the case $m = c = 1$ of Section 5, where the shape equation (5.1) reads $D_n + 1 = (a_n - 1)(n + 1)$, each multiplier is determined by its predecessor; we call such an orbit *forced*. At index n the numerator of the next multiplier is

$$\mathrm{num}(n, a) = (n + 1)a^2 - (n + 2)a + (n + 3),$$

the forced numerator, and the divisor is $n + 2$. The orbit survives a step when that division is exact, giving a survival predicate, the survival predicate. Example 5.1 is the forced orbit from $a = 3$ read this way: $\mathrm{num}(0, 3) = 6$ is divisible by 2 and gives $a_1 = 3$, while $\mathrm{num}(1, 3) = 13$ is not divisible by 3, so the orbit stops there. Deciding survival by iteration is expensive because the orbit grows doubly exponentially, and it is unnecessary: survival over a finite horizon depends on the initial value only through a factorial residue.

Theorem B.1 (factorial residue reduction)**.** *For all h and all integers $a \equiv b \pmod{(h+1)!}$, the orbit from a survives h forced updates exactly when the orbit from b does.*

Proof. Let $M(0, i) = 1$ and $M(h + 1, i) = (i + 2)M(h, i + 1)$, an ascending factorial with $M(h, 0) = (h + 1)!$. The numerator is a polynomial with integer coefficients, so congruences transfer; reducing the modulus gives divisibility by $i + 2$ for one exactly when for the other, and cancelling that common factor from both values and modulus leaves the inductive hypothesis at $M(h, i + 1)$. □

Formalised as the factorial residue reduction, over the shrinking-modulus transport, the polynomial congruence, and the exact-division cancellation; the modulus identifications are the ascending-factorial form and the factorial value at the initial index.

At $h = 1$ the modulus is $2! = 2$, and surviving one update means $2 \mid a^2 - 2a + 3$, which holds exactly for odd a: for instance $\mathrm{num}(0,3) = 6$ but $\mathrm{num}(0,4) = 11$. So one step of survival is decided by the parity of a alone, which is Theorem B.1 at its smallest nontrivial horizon.

The search this supported is superseded. Running it over initial states below 5000 produced forced prefixes of length 17 and no longer, which was evidence and not a proof; Theorem 5.2 now excludes the constant-negative case outright, for every seed and at every scale. The reduction is retained because it is exact, and because the shrinking-modulus technique transfers to any forced orbit whose step is a polynomial division.

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[7] T. F. Bloom, *Erdős Problem #243*, `erdosproblems.com/243`, accessed 28 July 2026 (page displays "last edited 21 January 2026"). The current record labels the problem open, cites `[ErGr80, p. 64]` and `[Er88c, p. 105]`, and explicitly describes its status as the website owner's present assessment rather than a literature-completeness guarantee.

[8] The Formal Conjectures Authors, *FormalConjectures.ErdosProblems.243*, Lean source at commit `f776d2f`, 2025, accessed 28 July 2026. Its $\mathbb{Q}$-valued `Summable` hypothesis encodes rationality, its indexing is zero-based, and its proof ends in `sorry`.

Statement	Status	Treatment here
Problem #243 as stated	Open	Not proved.
Defect identity	Proved here	Theorem 3.1, an identity in $\mathbb{Z}[a, a', D, C]$.
Eventual vanishing forces the recurrence	Proved here	Theorem 3.4.
Zero is absorbing	Proved here	Theorem 3.5, under strict centring.
Nonnegative E	Proved here	Theorem 4.1, by descent.
Constantly negative E	Excluded here	Theorem 5.2, at any magnitude and scale.
Periodic negative magnitude	Excluded here	Theorem 6.1, in the regime $e_n < a_n$ and with positive drift $M > 0$.
Coprime-modulus barrier	Proved here	Theorem 7.3.
Bounded negative part	Excluded here; conditional	Theorem 8.1; see its hypotheses.
Normalised vanishing alone	Gcd changes are sparse	Proposition 7.7; arbitrarily late finite constant blocks, not eventual constancy.
Finite normalised negative mass	Excluded here; conditional	Theorem 9.1, Section 9; normalised vanishing remains an input.
Factorial residue reduction	Proved here; superseded for its original purpose	Theorem B.1, Appendix B.
Normalised vanishing	Assumed here; supplied by prior art	Section 8; derived in [6], see Section 1. The separate strict-centring hypothesis retained in the Lean-matching statement follows eventually by taking $K = 1$.
Unbounded, divergent-mass negative excursions	Necessary conditions proved here; orbit-level case open	Section 10: Proposition 10.1 gives conditions any counterexample must satisfy, not a separate open problem; Problem 10.2 is the surviving orbit-level obstruction.
Erdős–Straus criterion; Duverney	Proved elsewhere	Cited, not formalised; the Erdős–Straus hypothesis is implied by that of Theorem 4.1.
State system equals reciprocal tails	Not formalised	Section 2.