Full support and positive density in the zero-temperature SK model

Will Cook · An account of the quantitative proof of OpenAI (2026a) · 19 November 2026

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Abstract. For the pure, zero-field Sherrington–Kirkpatrick model at zero temperature, the Stieltjes measure of the minimiser of the Parisi functional has full support on $[0,1)$ and a smooth density that is strictly positive on $(0,1)$. Full support and smoothness are due to H.-B. Chen. Strict positivity, which they do not imply, is due to OpenAI. We give an account of OpenAI's proof, in which one coercive inequality between spatial moments of the Parisi solution is used twice: on a constant interval it excludes gaps, and at the minimiser it bounds the density from below. The inequality requires no minimality, which is what the second use needs. We compare it with the crossing argument of Chen's proof, print the rational identity behind it in full, and work out the simplest finite-step case, in which the inequality holds with constant $2/5$ and no better. The shape inequalities and the regularity theory are cited.

1. The model and the result

Three statements about the zero-temperature Parisi minimiser have to be kept apart. Its support is infinite (Auffinger, Chen and Zeng (2020), Theorem 1). Its support is the whole interval, and its measure has a smooth density (H.-B. Chen (2026), Theorem 1.3). Its density is strictly positive in the interior (OpenAI (2026a), Corollary 6.4). None implies the next: the measure $\sum_{k\ge1} 2^{-k}\delta_{1-2^{-k}}$ has infinitely many atoms and many gaps, and a smooth density of full support can vanish at a point. This paper explains the proof of the third statement and its relation to the proof of the second. We fix the notation first.

For independent standard Gaussian couplings J_{ij}, the pure, zero-field Sherrington–Kirkpatrick (SK) Hamiltonian is

$$H_n(\sigma) = n^{-1/2}\sum_{i<j} J_{ij}\sigma_i\sigma_j, \qquad \sigma\in\{-1,1\}^n.$$

With the maximisation convention, its limiting ground-state energy per spin is

$$P_* := \lim_{n\to\infty} n^{-1}\mathbb{E}\max_{\sigma} H_n(\sigma).$$

The overlap of two configurations is $R(\sigma,\tau)=n^{-1}\sum_i \sigma_i\tau_i$, and it determines the energy covariance, $\mathbb{E}[H_n(\sigma)H_n(\tau)] = nR^2/2 - 1/2$. We use the covariance convention $\xi(t)=t^2/2$, so $\xi''=1$. Adding to every energy the same independent centred Gaussian of variance $1/2$ gives covariance $n\xi(R)$ and leaves the expected maximum unchanged, which is the normalisation of the ground-state Parisi formula of Auffinger and Chen (2017), Theorem 1. The model has no higher-order interactions and no external field.

Let $\mathcal{U}$ consist of the nonnegative, nondecreasing, right-continuous functions $\gamma:[0,1)\to[0,\infty)$ satisfying $\int_0^1\gamma(t)\,dt<\infty$. An element of $\mathcal{U}$ is an order parameter, and its locally finite Stieltjes measure is defined by

$$\mu_\gamma([0,t]) = \gamma(t), \qquad 0\le t<1.$$

Thus $\mu_\gamma((s,t]) = \gamma(t)-\gamma(s)$. The measure may have an atom of size $\gamma(0)$ at zero, and its total mass need not be finite or equal to one. Integrability allows $\gamma(t)\to\infty$ as $t\to1$, so the mass near one can be infinite, and γ need not be the distribution function of a probability measure. The point

0 lies in the support whenever $\gamma(t) > 0$ for all $t > 0$, even if $\gamma(0) = 0$. Support statements refer to the relative topology of $[0, 1)$. A gap is a relatively open interval of positive length and zero mass, and γ is constant on it.

For $\gamma \in \mathcal{U}$ the Parisi functional is defined through the solution $\Phi = \Phi_\gamma$ of the Parisi equation with terminal value $|x|$:

$$\Phi_t = -\tfrac{1}{2}(\Phi_{xx} + \gamma(t)\Phi_x^2), \qquad \Phi(1, x) = |x|, \qquad \mathcal{P}(\gamma) = \Phi_\gamma(0, 0) - \tfrac{1}{2}\int_0^1 t\gamma(t)\, dt.$$

The terminal value $|x|$ is the zero-temperature value of one spin in a field x. For an integrable order parameter, the solution is defined by its stochastic control representation, or equivalently by approximation from bounded monotone coefficients. OpenAI (2026a), §1 and Lemmas 2.1–2.2 give the exact control definition and the stability theorem. The Parisi formula at zero temperature states that $P_* = \min_{\gamma \in \mathcal{U}} \mathcal{P}(\gamma)$ (Auffinger and Chen (2017), Theorem 1 and the proof of Lemma 3), and the minimiser is unique (Chen, Handschy and Lerman (2018), Theorem 4).

Write $u = \Phi_x$ and $a = \Phi_{xx}$. Let B be standard Brownian motion, and define the controlled diffusion X and the consistency function q by

$$dX_t = \gamma(t)u(t, X_t)\, dt + dB_t, \qquad X_0 = 0, \qquad q(t) = \mathbb{E}[u(t, X_t)^2].$$

The SDE has a unique strong solution. Unless an endpoint limit is justified explicitly, all stochastic calculations below take place on compact time intervals ending before one.

At positive temperature, Lopatto (2026), Theorem 1.1 proved full replica symmetry breaking throughout the low-temperature phase: at each inverse temperature $\beta > 1$ the Parisi measure has support $[0, q_\beta]$, with $q_\beta < 1$. At zero temperature, full support and a smooth nonnegative density were proved in H.-B. Chen (2026), Theorem 1.3, together with the endpoint estimate $\beta^{-2} < 1 - q_\beta < 2\beta^{-2}$ (Chen (2026), Theorem 1.2).

Theorem 1 (full support; H.-B. Chen (2026), Theorem 1.3; OpenAI (2026a), Theorem 1.1). If $\gamma \in \mathcal{U}$ minimises $\mathcal{P}$, then

$$\operatorname{supp} \mu_\gamma = [0, 1), \qquad \gamma(t_2) > \gamma(t_1) \quad (0 \le t_1 < t_2 < 1).$$

For the solution of the displayed SDE and every $0 \le t < 1$,

$$q(t) = t, \qquad \mathbb{E}[a(t, X_t)^2] = 1.$$

Corollary 2 (smooth positive density; OpenAI (2026a), Corollary 6.4). Under the hypotheses of Theorem 1, $\gamma(0) = 0$ and $\gamma \in C^\infty([0, 1))$, with derivatives at zero taken from the right, so that $\mu_\gamma(dt) = \gamma'(t)\, dt$. Moreover

$$\gamma'(t) \ge \frac{\gamma(t)^2}{2000} > 0 \qquad (0 < t < 1).$$

Smoothness and $\gamma(0) = 0$ are already in Chen's theorem. The strict lower bound is new in OpenAI's corollary, which states it in the form (10) below, and the explicit bound $\gamma^2/2000$ is deduced here. Only nonnegativity of the density at zero is asserted. Integrability allows γ to grow near one. Zero field gives $u(0, 0) = 0$, and the curvature calculation uses $\xi'' = 1$, so both statements concern only the pure SK covariance at zero field.

The two proofs. Both proofs start from the first variation of $\mathcal{P}$. At the minimiser it gives $q(s) = s$ and $q'(s) \le 1$ at every point s of the support, the consistency and stability conditions. On an interior gap the order parameter is a positive constant c, and a contradiction with these endpoint conditions must come from the behaviour of q inside the gap. Let v, w, z be the second, third and fourth spatial derivatives of $c\Phi$ along the diffusion. Then

$$c^2 q'' = \mathbb{E}[w^2 - 2v^3], \qquad c^2 q''' = \mathbb{E}[z^2 - 12vw^2 + 6v^4].$$

Chen's proof establishes the crossing property $q''(t) = 0 \Rightarrow q'''(t) > 0$ on the gap (Chen (2026), Proposition 4.2), in parallel with the positive-temperature argument of Lopatto (2026), v3, Proposition 9.1. Hence q'' changes sign at most once, from negative to positive. The endpoint conditions make a weighted integral of q'' over the gap vanish, which fixes this sign pattern, and a second integration then contradicts stability at the left endpoint (§2).

OpenAI's proof establishes the unconditional bound

$$c^2 q'''(t) \ge \frac{1}{1000}\, \mathbb{E}\, v(t, X_t)^4 > 0$$

on every positive constant interval of every order parameter (OpenAI (2026a), Proposition 5.1). The polynomial $z^2 - 12vw^2 + 6v^4$ takes negative values, so the bound is not pointwise. It follows from an exact polynomial identity with rational coefficients which, after one integration by parts against the density of X_t, writes the expectation as a sum of nonnegative terms and $\kappa\, \mathbb{E} v^4$ with $\kappa > 1/1000$.

The bound is stronger than the crossing property in two respects, and the proof of Corollary 2 uses both. It needs no hypothesis on q'', and it is a non-strict inequality with an explicit constant between spatial moments at one time. If γ is replaced by the constant $\gamma(t)$ on a short interval around t, the bound holds for the modified order parameter at time t, although that order parameter does not minimise, and it passes to the limit as the interval shrinks. This gives Proposition 5 of §4: at each $0 < t < 1$ with $\gamma(t) > 0$,

$$\mathbb{E}\big[\Phi_{xxxx}^2 - 12\gamma\, \Phi_{xx}\Phi_{xxx}^2 + 6\gamma^2 \Phi_{xx}^4\big] \ge \frac{\gamma(t)^2}{1000}\, \mathbb{E}\, \Phi_{xx}^4.$$

For a smooth order parameter the left side equals $q''' + 2\gamma' \mathbb{E} a^3$. At the minimiser $q''' = 0$, and Corollary 2 follows.

Organisation. Section 2 derives the consistency and stability conditions and proves Theorem 1 from the curvature bound, stated there as Lemma 3. Section 3 proves the main estimate. It works out an explicitly solvable case first, then states the shape inequalities, the rational identity in full (equation (5) and Table 1) and the passage to integrable order parameters, and ends with the relation to the inequalities of Lopatto and Chen. Section 4 proves Proposition 5 and Corollary 2. Section 5 states the consequences for optimisation, which use Theorem 1 only. The final section records what the Lean formalisation covers.

Sections 2–4 assume Itô calculus and the regularity theory of the Parisi equation. Proofs that we omit are in OpenAI's manuscript at the places cited, and in *From Parisi full support to deterministic QAOA circuits: A long reasoning record*, the companion edition of 19 November 2026, cited below as the companion.

2. The variational conditions and the exclusion of gaps

For $\gamma \in \mathcal{U}$, the solution Φ is even, convex and one-Lipschitz in space. Its spatial derivatives of positive order are bounded and continuous on each strip $[0,T] \times \mathbb{R}$, $T < 1$, and the PDE holds for almost every time. The compact-strip regularity and stability results of OpenAI (2026a), Lemmas 2.1–2.2 justify the Itô calculations below even when γ has jumps. These facts go back to Jagannath and Tobasco (2016) at positive temperature and to Auffinger and Chen (2017), Theorem 2 and Corollary 2 and Chen, Handschy and Lerman (2018), Proposition 2 and Theorem 5 at zero temperature.

We fix the minimiser. Itô's formula cancels the drift of $u(t, X_t)$:

$$du(t, X_t) = a(t, X_t)\, dB_t, \qquad q(0) = 0, \qquad q'(t) = \mathbb{E}[a(t, X_t)^2].$$

In particular, q' is continuous across jumps of the coefficient. When γ is replaced by $\gamma + \varepsilon\eta$ and $\varepsilon \downarrow 0$, the functional changes at the rate

$$D_+\mathcal{P}(\gamma)[\eta] = \tfrac{1}{2} \int_0^1 \eta(t)(q(t) - t)\, dt, \tag{1}$$

for every integrable direction η for which $\gamma + \varepsilon\eta \in \mathcal{U}$ for all sufficiently small $\varepsilon \geq 0$ (see OpenAI (2026a), Lemma 6.1, and Chen, Handschy and Lerman (2018), Propositions 3–4 for the earlier zero-temperature form).

Adding a unit atom at s raises the cumulative function at every $t \geq s$, so its direction is $\eta = \mathbf{1}_{[s,1)}$. Minimality makes the corresponding tail integral nonnegative:

$$G(s) := \int_s^1 (q(t) - t)\, dt \geq 0.$$

Mass can also be removed. Let $\nu = \mu_\gamma|_K$ for a compact $K \subset [0,1)$. For $0 \leq \varepsilon \leq 1$, the measure $\mu_\gamma - \varepsilon\nu = (1 - \varepsilon \mathbf{1}_K)\mu_\gamma$ is nonnegative, and its cumulative function is $\gamma - \varepsilon\nu([0,\cdot]) \in \mathcal{U}$. Minimality gives

$$0 \leq D_+\mathcal{P}(\gamma)[-\nu([0,\cdot])] = -\tfrac{1}{2} \int_K G(s)\, d\mu_\gamma(s).$$

The bounds $|q - t| \leq 1$ and $\mu_\gamma(K) < \infty$ justify the change in the order of integration. Since $G \geq 0$, it vanishes almost everywhere for μ_γ, and by continuity it vanishes on $S = \operatorname{supp} \mu_\gamma$. No probability constraint is involved, since $\mathcal{U}$ carries none.

At a support point $s > 0$, the function G has a minimum. Hence

$$G'(s) = s - q(s) = 0, \qquad G''(s) = 1 - q'(s) \geq 0, \quad \text{so} \quad q(s) = s, \quad q'(s) \leq 1. \tag{2}$$

If $0 \in S$, then $G(0) = 0$, and zero field gives $q(0) = 0$ independently. Consequently $G(h) = \frac{1}{2}(1 - q'(0))h^2 + o(h^2)$, and nonnegativity gives the same slope bound at zero. These are the consistency and stability conditions, in the form of OpenAI (2026a), Lemma 6.2 (compare Chen (2026), Proposition 3.2).

Lemma 3 (curvature on a constant interval; OpenAI (2026a), Proposition 5.1). Let $\gamma \in \mathcal{U}$, with its diffusion and consistency function q. If $\gamma = c$ on an open interval $I \subset (0,1)$, then for $t \in I$

$$q'''(t) \geq \frac{c^2}{1000} \mathbb{E}[a(t, X_t)^4] > 0 \quad \text{if } c > 0. \tag{3}$$

If $c = 0$, then $q''(t) = \mathbb{E}[\Phi_{xxx}(t, X_t)^2] > 0$.

The lemma holds for every order parameter, minimising or not. Section 3 is devoted to its proof.

Proof of Theorem 1, assuming Lemma 3. We treat the intervals of the complement of S by type.

Suppose first that (ℓ, ℓ') is a gap with endpoints in S and $\ell' < 1$. The coefficient is a positive constant there, because support at ℓ and the absence of mass immediately to its right force $\mu_\gamma([0, \ell]) > 0$. Strict convexity of q' and (2) give

$$q'(t) < \frac{\ell' - t}{\ell' - \ell} q'(\ell) + \frac{t - \ell}{\ell' - \ell} q'(\ell') \le 1 \qquad (\ell < t < \ell').$$

Integrating contradicts $q(\ell') - q(\ell) = \ell' - \ell$.

An initial gap $[0, \ell')$ with $0 \notin S$ and $\ell' \in S$ carries no mass, hence $\gamma = 0$ there. Its right endpoint gives $q(\ell') = \ell'$ and $q'(\ell') \le 1$, and zero field gives $q(0) = 0$. The second part of Lemma 3 gives $q'' > 0$, hence $q'(t) < q'(\ell') \le 1$, and the same integral contradiction follows without assuming $G(0) = 0$. A gap $(0, \ell')$ with $0 \in S$ is already covered by the first case.

A terminal gap $(\ell, 1)$ with nonempty support has $\ell \in S$ and a positive constant c. Suppose that $q(t) \ge t$ for some $\ell < t < 1$. Strict convexity of q' and $q'(\ell) \le 1$ give

$$t - \ell \le \int_\ell^t q'(s)\, ds < \frac{t - \ell}{2} (q'(\ell) + q'(t)),$$

so $q'(t) > 1$. Convexity then gives $q'(s) > q'(t) > 1$ for $s > t$, and consequently $q(s) > t + (s - t)q'(t) > 1$ for $s < 1$ sufficiently close to one, contrary to $q \le 1$. Thus $q(t) < t$ on $(\ell, 1)$, and $G(\ell) < 0$ contradicts minimality. This case uses the conditions (2) only at the left endpoint.

If S is empty, then $\gamma = 0$. Lemma 3 makes q strictly convex, and $q(0) = 0$ and $q \le 1$ give $q(t) < (t/b)q(b) \le t/b$ for $0 < t < b < 1$. Letting $b \uparrow 1$ gives $q(t) \le t$. Strict convexity applied once more with $t < b < 1$ now gives $q(t) < (t/b)q(b) \le t$, so $G(0) < 0$.

These cases exhaust the complement of the relatively closed set S. Consequently $S = [0, 1)$, the conditions (2) give $q(t) = t$, and differentiation gives $\mathbb{E}a^2 = 1$, including the right-hand value at zero. Moreover, $\gamma(t_2) - \gamma(t_1) = \mu_\gamma((t_1, t_2]) > 0$. □

Remarks on the endpoints. The exclusion of a terminal gap was noted in Auffinger, Chen and Zeng (2020), Remark 7. The proof of OpenAI (2026a), Lemma 6.3 gives the stronger boundary estimate $1 - q(t) \ge C\sqrt{1 - t}$ for any order parameter constant near one, and Chen (2026), Proposition 4.5, equations (4.27)–(4.38) gives the square-root asymptotic under the terminal-gap hypothesis. At the left endpoint, Chen takes $0 \in S$ from Chen, Handschy and Lerman (2018) (see Chen (2026), Proposition 3.2 and Remark 4.4). The proof above treats an initial gap directly through the heat equation, for which antecedents appear in Chen, Handschy and Lerman (2018), Theorem 6.

Comparison with the crossing argument. In Chen (2026), Proposition 4.3 an internal gap (ℓ, ℓ') is excluded as follows. The crossing property shows that q'' has at most one zero in the gap. The equalities $G(\ell) = G(\ell') = 0$ and $q(\ell) = \ell$, $q(\ell') = \ell'$ give $\int_\ell^{\ell'} (t - \ell)(\ell' - t)\, q''(t)\, dt = 0$, so q'' is negative and then positive, and a second integration yields $q(\ell') - q(\ell) < (\ell' - \ell)q'(\ell+) \le \ell' - \ell$. Only the stability condition at the left endpoint is used. With (3) available, strict convexity of q' replaces the analysis of the sign changes of q'', at the price of using stability at both endpoints.

3. The coercive inequality on a constant interval

On an interval where $\gamma = c > 0$, put $r = c\Phi_x$, $v = r_x$, $w = r_{xx}$ and $z = r_{xxx}$. The PDE becomes $r_t = -\frac{1}{2}r_{xx} - rr_x$, and the diffusion has spatial generator $\frac{1}{2}\partial_{xx} + r\partial_x$. Differentiating in space and applying Itô's formula gives the drift and diffusion pairs $(0, v)$, $(-v^2, w)$ and $(-3vw, z)$ for r, v, w, respectively. All expectations in this section evaluate spatial functions at (t, X_t). Starting from $c^2 q' = \mathbb{E}v^2$, Itô's formula gives

$$\begin{aligned} c^2 q'' &= \mathbb{E}[w^2 - 2v^3], \\ c^2 q''' &= \mathbb{E}[z^2 - 12vw^2 + 6v^4]. \end{aligned} \tag{4}$$

For example, the drifts of w^2 and v^3 are $z^2 - 6vw^2$ and $3vw^2 - 3v^4$, respectively. Lemma 3 for $c > 0$ is therefore the inequality $\mathbb{E}[z^2 - 12vw^2 + 6v^4] \geq \mathbb{E}v^4/1000$. The equation for r is invariant under $r(t, x) \mapsto \lambda r(\lambda^2 t, \lambda x)$, which multiplies v, w, z by $\lambda^2, \lambda^3, \lambda^4$, and the polynomial is homogeneous of weight eight for this scaling.

3.1 An explicitly solvable case

The proof of the inequality works with finite-step coefficients and the smooth terminal value $M^{-1}\log(2\cosh Mx)$, where M bounds the coefficient. Its simplest instance is a single step with $c = M$, and by the scaling we may take $c = M = 1$. Then

$$\Phi(t, x) = \log(2\cosh x) + \tfrac{1-t}{2}, \qquad r = \tanh x,$$

so the drift does not depend on time, and X_t has the law $\frac{1}{2}N(t, t) + \frac{1}{2}N(-t, t)$. With $\theta = \tanh^2 x$ we have $v = 1 - \theta$, $w^2 = 4\theta(1 - \theta)^2$ and $z = (1 - \theta)(6\theta - 2)$, hence

$$z^2 - 12vw^2 + 6v^4 = (1 - \theta)^2 (90\theta^2 - 84\theta + 10).$$

This is negative for $(7 - 2\sqrt{6})/15 < \theta < (7 + 2\sqrt{6})/15$, that is, for $0.140 < \theta < 0.793$, and it equals $-19/8$ at $\theta = 1/2$. At $t = 1/2$ the diffusion lies in the negative region with probability 0.56. Convexity of Φ, which gives $v \geq 0$, therefore does not control the mixed term.

The expectation is nevertheless positive. The density of X_t is

$$f(t, x) = \frac{\cosh x}{\sqrt{2\pi t}}\, e^{-x^2/(2t) - t/2},$$

and $\rho(t) := \mathbb{E}[z^2 - 12vw^2 + 6v^4]/\mathbb{E}v^4$ has the limits

$$\lim_{t \downarrow 0} \rho(t) = 10, \qquad \lim_{t \to \infty} \rho(t) = \frac{\int (16\operatorname{sech}^3 x - 96\operatorname{sech}^5 x + 90\operatorname{sech}^7 x)\, dx}{\int \operatorname{sech}^7 x\, dx} = \frac{\pi/8}{5\pi/16} = \frac{2}{5}.$$

The first holds because $X_0 = 0$, and the second by dominated convergence, since $\sqrt{2\pi t}\, e^{t/2} f(t, x) \to \cosh x$.

Moreover $\rho(t) > 2/5$ for every $t > 0$. Put

$$F(x) = 16\operatorname{sech}^3 x - 96\operatorname{sech}^5 x + \tfrac{448}{5}\operatorname{sech}^7 x = \tfrac{1}{5}\cosh x\,(1 - \theta)^2 (448\theta^2 - 416\theta + 48),$$

so that $\rho(t) - 2/5$ is a positive multiple of $\int_0^\infty F(x) e^{-x^2/(2t)}\, dx$. The quadratic has two roots in $(0, 1)$, so F is positive, then negative, then positive on $(0, \infty)$, and the integrals of powers of sech give $\int_0^\infty F\, dx = 0$ and $\int_0^\infty x^2 F\, dx = -52\pi/225$. Hence $G(x) = \int_0^x F$ is positive and then negative, with

$G(\infty) = 0$, and $K(x) = \int_0^x yG(y)\,dy$ increases and then decreases to $K(\infty) = -\frac{1}{2}\int_0^\infty x^2 F\,dx > 0$, so $K > 0$ on $(0, \infty)$. Two integrations by parts give $\int_0^\infty F(x)e^{-x^2/(2t)}\,dx = t^{-2}\int_0^\infty K(x)\,x\,e^{-x^2/(2t)}\,dx > 0$. For a single step of height $c = M$ the ratio at time t is $\rho(c^2 t)$. Over this family, with $0 < t < 1$, the best constant in the integrated inequality, (6) of §3.3, is therefore $2/5$, not attained, against the $1/1000$ of the general proof. This concerns the auxiliary inequality and not the density of the minimiser. Numerically $\rho(1/2) = 2.39$, and this value is decomposed term by term in §3.3.

The same pointwise obstruction occurs for the terminal value $|x|$. For $\gamma \equiv c$ and $\tau = 1 - t$, an expansion at small c gives $z^2 - 12vw^2 + 6v^4 = -12c^3 a_0^3/\tau + O(c^4) < 0$ at $x = \sqrt{\tau}$, where $a_0 = \sqrt{2/(\pi\tau)}\,e^{-1/2}$ (companion, §4.1).

3.2 The two heat equations and the shape inequalities

We now take a positive nondecreasing finite-step coefficient γ_{step}, that is, a function constant on each interval of a finite partition, and the terminal value $M^{-1}\log(2\cosh Mx)$ with $M \geq \|\gamma_{\text{step}}\|_\infty$. On a step of height c, set $h = e^{c\Phi}$. Then $h_t = -h_{xx}/2$ and $r = h_x/h$, the Cole–Hopf transformation. The same exponential removes the drift from the equation for the density f of X_t. Indeed, writing $f = hg$ gives

$$\begin{aligned} f_t &= \tfrac{1}{2}(hg)_{xx} - (h_x g)_x \\ &= -\tfrac{1}{2}h_{xx}g + \tfrac{1}{2}hg_{xx} = h_t g + \tfrac{1}{2}hg_{xx}. \end{aligned}$$

Comparison with $f_t = h_t g + hg_t$ yields $g_t = g_{xx}/2$. Thus $f = hg$ with h a solution of the backward heat equation and g a solution of the forward one. The factor g is the transformed density of Lopatto (2026), Proposition 7.1 and the function Q of Chen (2026), §3.2. Its score is $s = -\partial_x \log g$, so that $f_x/f = r - s$. On the first step $s = x/t$.

The functions r and s are odd and positive on $x > 0$. On $x > 0$, at every time of every step, OpenAI (2026a), Lemmas 3.2–3.3 prove

$$\begin{gathered} v > 0, \qquad w \leq 0, \qquad H := \frac{w}{2r} + v \leq 0, \qquad H_x \geq 0, \\ s_x \geq \frac{1}{t}, \qquad s_{xx} \leq 0, \qquad j := s_x - \frac{s}{r}\,v = r\,\partial_x\Big(\frac{s}{r}\Big) \geq 0. \end{gathered}$$

Thus r and s are increasing and concave, and the ratio s/r of the two scores is nondecreasing. Since $r_t = -\frac{1}{2}w - rv = -rH$, we have $H = -\partial_t \log \Phi_x$, and $H \leq 0$ says that the gradient $\Phi_x(t, x)$ at a fixed $x > 0$ is nondecreasing in t within a step.

Each sign is propagated within a step by a parabolic comparison and checked at the jumps. For example, in backward time τ the function w solves the linear equation $w_\tau = \frac{1}{2}w_{xx} + rw_x + 3vw$. It vanishes at $x = 0$ and is nonpositive at the terminal time, so the maximum principle on $x > 0$ gives $w \leq 0$. The equation for H has the first-order coefficient $v/r + r = x^{-1} + O(x)$, and it becomes a regular equation for the radial function $H(|y|)$ on $\mathbb{R}^3$, to which the same principle applies. For the ratio, concavity of r gives $j \geq 0$ on the first step, where $s = x/t$. At a forward jump with $k = c_+/c_- \geq 1$, continuity of Φ and f gives

$$r_+ = kr_-, \qquad s_+ = s_- + (k-1)r_-, \qquad j_+ = (s_x)_+ - \frac{s_+}{r_+}(r_x)_+ = j_-.$$

Thus j is unchanged at jumps of the coefficient. The quotients w/r and s/r extend smoothly through zero because r, s, w are odd and $r_x(0) > 0$. The ratio inequality is the one that does not

appear in the crossing arguments (§3.5). It also follows from Zhou (2026), Appendix E.3, (E.37) and (E.41), after rescaling the smoothed finite-step problem (companion, §4.3).

In the case of §3.1, $H \equiv 0$, $s = x/t$ and $j = t^{-1}(1 - 2x/\sinh 2x)$. The backward inequalities for H are equalities there, and H measures the departure of a general step from the profile tanh.

3.3 The rational identity

The inequalities are used through nonnegative even quantities. Following OpenAI (2026a), §3.4, set

$$R = r^2, \quad K = rs, \quad b = -\frac{w}{r}, \quad B = b - 2v = -2H, \quad S = z + bv - 2Rb = 2rH_x, \quad L = -rs_{xx},$$

together with v and j. All eight are nonnegative on the line, and $Rb^2 = w^2$. The letters are those of OpenAI (2026a), so that the identity (5) below can be compared with its Table 1. Within §§3.3–3.5 they replace the earlier meanings of B, S, R and K, and Q and T denote the polynomials defined below. For the scaling at the beginning of this section, R, K, v, b, j, B have weight two and z, S, L weight four. For $\mathcal{D} = r\partial_x$ one has

$$\mathcal{D}(R, K, v, b, j) = (2Rv,\ 2vK + Rj,\ -Rb,\ -z - bv,\ -jv + Kb - L),$$

so $\mathcal{D}$ maps polynomials in R, K, v, b, j to polynomials in R, K, v, b, j, z, L. For such a polynomial F put

$$\mathcal{W}(F) = \mathcal{D}F + (v + R - K)F.$$

Since $f_x/f = r - s$, we have $f\,\mathcal{W}(F) = \partial_x(rfF)$. The smooth extensions avoid an interior boundary term at zero, and the differentiated Gaussian tail estimates give $rfF \to 0$ at both spatial infinities. Hence $\mathbb{E}_f \mathcal{W}(F) = 0$, where $\mathbb{E}_f$ denotes integration against $f(x)\,dx$ (OpenAI (2026a), Lemma 3.4).

The problem is thus to find a polynomial F such that $z^2 - 12vw^2 + 6v^4 - \mathcal{W}(F)$ is a sum of squares and of products of the nonnegative quantities, with a positive multiple of v^4 among them. By homogeneity F is a cubic form in R, K, v, b, j. The terms containing z are arranged first. The only rule of $\mathcal{D}$ that produces z is $\mathcal{D}b = -z - bv$, so the coefficient of z in $\mathcal{W}(F)$ is $-\partial_b F$. Choose

$$\begin{aligned} Q &= \tfrac{1}{13}vb + \tfrac{53}{24}v^2 + \tfrac{19}{16}Kv - \tfrac{19}{8}Rb + \tfrac{73}{48}Rv, \\ T &= v(\tfrac{2331}{10000}R + \tfrac{87}{2500}j), \end{aligned}$$

and let F_2 satisfy $\partial_b F_2 = 2Q + T$:

$$\begin{aligned} F_2 = {} & (\tfrac{1}{13}v - \tfrac{19}{8}R)b^2 + b(\tfrac{53}{12}v^2 + \tfrac{19}{8}Kv + \tfrac{98243}{30000}Rv + \tfrac{87}{2500}jv) \\ & + \tfrac{4663}{5000}v^2 j + \tfrac{6853}{10000}v^3 - \tfrac{1051}{400}Kv^2 - \tfrac{1073}{500}Rv^2. \end{aligned}$$

Then $z^2 - (z + Q)^2 - ST - \mathcal{W}(F_2)$ contains no z. The last line of F_2 does not involve b and is free for the remaining terms.

Lemma 4 (rational identity; OpenAI (2026a), Lemma 4.1). As polynomials over $\mathbb{Q}$ in R, K, v, b, j, z, L, with $B = b - 2v$ and $S = z + bv - 2Rb$,

$$\begin{aligned} z^2 - 12Rvb^2 + 6v^4 = {} & \mathcal{W}(F_2) + (z + Q)^2 + ST + \tfrac{4663}{5000}Lv^2 + \tfrac{87}{2500}Lbv \\ & + \tfrac{337}{10000}Rj(b - 7v)^2 + \tfrac{183}{2500}RB(b - \tfrac{7}{3}v)^2 + \tfrac{411}{10000}Kv(b - \tfrac{9}{2}v)^2 \\ & + \tfrac{11461}{5000}vB(K + \tfrac{1}{4}R - \tfrac{1}{7}v)^2 + \tfrac{647}{10000}v^2(b - \tfrac{14}{5}v - \tfrac{13}{4}K + \tfrac{4}{3}R)^2 + P_{\mathrm{rem}}, \end{aligned} \tag{5}$$

where P_{rem} is the polynomial in R, K, v, B, j with the 22 positive coefficients of Table 1.

Monomial	Coefficient	Monomial	Coefficient
R^2v^2	223/160000	R^2vB	4007/240000
R^2B^2	1097/120000	RKv^2	83/80000
$RKvB$	3343/24000	Rv^3	16259/23400000
Rv^2B	30179/5460000	Rv^2j	26/625
RvB^2	107/15000	$RvBj$	3/2500
RB^3	121/32500	RB^2j	11/10000
K^2v^2	579/20000	K^2vB	207/2500
Kv^3	911/975000	Kv^2B	349/420000
Kv^2j	5011/5000	KvB^2	133/130000
$KvBj$	87/2500	v^4	1661507/1521000000
v^3B	633799/621075000	v^2B^2	10657/1690000

Table 1. The coefficients of P_{rem} (OpenAI (2026a), Table 1).

The identity is verified by expanding both sides. Each term on the right other than $\mathcal{W}(F_2)$ is nonnegative, and each inequality of §3.2 enters at an identifiable place: $H_x \geq 0$ only through ST; concavity of the forward score only through the two terms with L, the second of which also uses $b \geq 0$; the ratio inequality through T, the square with prefactor Rj and five monomials of the table; and $H \leq 0$ through the prefactors RB, vB and the monomials containing B. Since $Rb^2 = w^2$, the left side is $z^2 - 12vw^2 + 6v^4$. Integration against f removes $\mathcal{W}(F_2)$ and keeps the coefficient of v^4:

$$\int_{\mathbb{R}} (z^2 - 12vw^2 + 6v^4) f \, dx \geq \kappa \int_{\mathbb{R}} v^4 f \, dx, \qquad \kappa = \frac{1661507}{1521000000} > \frac{1}{1000}. \tag{6}$$

This is OpenAI (2026a), Proposition 4.2.

In the case of §3.1 we have $B = S = L = 0$ and $b = 2v$, and (5) reduces to $(z + Q)^2$, the squares with prefactors Rj, Kv and v^2, and the eight monomials of the table without B. At $t = 1/2$ their expectations are 0.580, 0.036, 0.030, 0.135 and 0.115, with sum $0.896 = \mathbb{E}[z^2 - 12vw^2 + 6v^4]$, while $\kappa \, \mathbb{E}v^4 = 0.0004$. Most of the expectation is carried by the square $(z + Q)^2$, and κ is the coefficient guaranteed for all finite-step data.

OpenAI (2026a) does not say how the coefficients were found. The form of (5) is that of a sum-of-squares certificate modulo integration by parts: once the admissible products are fixed, the coefficients of F_2 and of the remainder enter linearly and the squares give a semidefinite constraint, so that the search is a finite-dimensional feasibility problem. Certificates of this form are constructed systematically for entropy inequalities in Jüngel and Matthes (2006). The identity itself is part of the Lean formalisation (see the final section).

3.4 Integrable order parameters

To obtain Lemma 3 for $\gamma \in \mathcal{U}$ with $\gamma = c > 0$ on I, fix $t \in I$ and choose positive monotone step coefficients γ_n equal to c on a fixed neighbourhood of t, with $\|\gamma_n - \gamma\|_1 \to 0$, and terminal smoothing parameters $M_n \geq \|\gamma_n\|_\infty$ with $M_n \to \infty$. Spatial derivatives through order four then converge uniformly on a fixed strip below one, and the diffusions driven by the same Brownian motion converge (see OpenAI (2026a), Lemmas 2.1–2.3 and Proposition 5.1). Only v, w, z remain

in (6), so no convergence of s, j or w/r is needed. Dividing by c^2 gives (3), and strictness follows from $\Phi_{xx} > 0$ on the interval. The approximants and their common strip bounds are specified in the companion, §4.5.

For $c = 0$ the equation on I is the heat equation and $q'' = \mathbb{E}\,\Phi_{xxx}^2$. Its strictness uses a positive diffusion density and the fact that Φ_{xxx} cannot vanish identically, since an even quadratic profile cannot differ from $|x|$ by a bounded function.

3.5 The slope coordinates of Lopatto and Chen

The crossing arguments record the shape of Φ on a constant interval $\gamma = m$ in the slope coordinate Φ_x (Lopatto (2026), v2, Proposition 3.6; Chen (2026), Propositions 3.10 and 3.11). With Φ_{xx} regarded as a function of the slope, $\mathsf{z} = -\frac{1}{2}\partial_{\Phi_x}\Phi_{xx}$, $\mathsf{K} = \mathsf{z}/\Phi_x - m$ and $\mathsf{J} = \partial_{\Phi_x}(\Phi_x\mathsf{K})$, they prove $\mathsf{K}, \partial_{\Phi_x}\mathsf{K}, \mathsf{J}, \partial_{\Phi_x}\mathsf{J} \geq 0$ and $\Phi_{xx}\,\partial_{\Phi_x}\mathsf{J} \leq 3\mathsf{z}\mathsf{J}$ for nonnegative slopes. Since $\partial_{\Phi_x} = \Phi_{xx}^{-1}\partial_x$, a direct calculation with $c = m$ gives, for $x > 0$,

$$\mathsf{K} = -\frac{H}{\Phi_{xx}} = \frac{cB}{2v}, \qquad \partial_{\Phi_x}\mathsf{K} = \frac{c^2\,(BRb - Sv)}{2rv^3}, \qquad \mathsf{J} = \frac{c\,(B(v^2 + Rb) - Sv)}{2v^3}.$$

Thus $\mathsf{K} \geq 0$ is the inequality $H \leq 0$, and $\partial_{\Phi_x}\mathsf{K} \geq 0$ is the upper bound $Sv \leq BRb$, which is not used in (5). The certificate uses instead the lower bound $S \geq 0$. For the smooth finite-step data of §3.2, to which both families apply, they give together

$$0 \leq Sv \leq BRb.$$

On the forward side, Chen writes Γ for q, Q for g, $\mathfrak{r}$ for s and H for $g_{xx}/g = s^2 - s_x$. For the finite-step data, the log-concavity of g and the monotonicity of g_{xx}/g on $x > 0$, which Chen (2026), Proposition 3.7 proves for the terminal value $|x|$, follow from $s \geq 0$, $s_x > 0$ and $s_{xx} \leq 0$, because $\partial_x(s^2 - s_x) = 2ss_x - s_{xx}$. The inequalities for $\partial_{\Phi_x}\mathsf{J}$ involve the fifth derivative of Φ and do not appear in (5), and the ratio inequality $j \geq 0$ does not appear in the slope-coordinate family. Whether either family implies the other is not addressed here. The inequalities for K and J originate in Lopatto (2026), v2, Proposition 3.6, and the transformed density appears in Lopatto (2026), v3, Proposition 7.1. The inequalities $H_x \geq 0$ and $j \geq 0$ are proved in OpenAI (2026a), Lemmas 3.2–3.3.

4. The coercive inequality at a fixed time and positivity of the density

For $\gamma \in \mathcal{U}$ and $0 < t < 1$ set

$$\mathcal{C}_\gamma(t) = \mathbb{E}\big[\Phi_{xxxx}^2 - 12\gamma(t)\,a\,\Phi_{xxx}^2 + 6\gamma(t)^2 a^4\big],$$

with the spatial factors evaluated at (t, X_t). On an interval where $\gamma = c$ this is $q'''(t)$, by (4). The definition involves no time derivative of γ. At a jump, $\gamma(t)$ is the right-continuous value, and the spatial derivatives of Φ are continuous in t.

Proposition 5 (coercive inequality at a fixed time). Let $\gamma \in \mathcal{U}$ and $0 < t < 1$, and suppose that $c = \gamma(t) > 0$. For its diffusion,

$$\mathcal{C}_\gamma(t) \geq \frac{c^2}{1000}\mathbb{E}[a(t, X_t)^4].$$

Neither minimality, nor continuity of γ at t, nor a constant neighbourhood of t is required.

The statement is implicit in OpenAI (2026a), proof of Corollary 6.4. We state it separately because it is the form in which the inequality is used at the minimiser.

Proof. For $0 < \delta < \min(t, 1-t)$ define

$$\gamma_\delta(s) = \begin{cases} c, & t-\delta \le s < t+\delta, \\ \gamma(s), & \text{otherwise.} \end{cases}$$

The inequalities $\gamma(t-\delta) \le c \le \gamma(t+\delta)$ make both joins nondecreasing, so $\gamma_\delta \in \mathcal{U}$, although it need not minimise $\mathcal{P}$. Moreover,

$$\|\gamma_\delta - \gamma\|_1 \le 2\delta\,[\gamma(t+\delta) - \gamma(t-\delta)] \longrightarrow 0.$$

We use a subscript δ for its value function, its derivative a_δ and its diffusion. Lemma 3 and (4) give at the fixed time t

$$\mathcal{C}_{\gamma_\delta}(t) \ge \frac{c^2}{1000}\mathbb{E}[a_\delta(t, X_t^\delta)^4]. \tag{7}$$

Choose $t < T < 1$. Lemma 2.1 of OpenAI (2026a) gives uniform convergence in space and uniform bounds for the second, third and fourth spatial derivatives of Φ_{γ_δ} on $[0, T]$, with the next spatial derivatives bounded as well. By Lemma 2.2, the diffusions coupled through the same Brownian motion satisfy $\sup_{s \le T} |X_s^\delta - X_s| \to 0$. The derivative bounds control evaluation along these paths, so the integrands in (7) converge and are uniformly bounded, and dominated convergence passes their expectations to the limit. These estimates depend only on the common L^1 bound on the coefficient and the fixed distance from time one, whatever the width of the constant interval. The result is $\mathcal{C}_\gamma(t) \ge \gamma(t)^2 \mathbb{E}a^4/1000$. □

The limit is taken in spatial moments at the single time t. It does not identify $q_\delta'''(t)$ with a derivative of the limiting consistency function, and no such identification holds: for the minimiser $q''' = 0$.

Full support alone does not force a pointwise positive density. For instance, $\gamma(t) = \int_0^t (s - \frac{1}{2})^2\,ds$ has full Stieltjes support but $\gamma'(\frac{1}{2}) = 0$.

Proof of Corollary 2. All spatial factors in the expectations below are evaluated at (t, X_t). Since $da = -\gamma a^2\,dt + \Phi_{xxx}\,dB$, Itô's formula applied to a^2 gives, for almost every t, $0 = \frac{d}{dt}\mathbb{E}a^2 = \mathbb{E}\,\Phi_{xxx}^2 - 2\gamma\mathbb{E}a^3$, where the left side vanishes because full support gives $\mathbb{E}a^2 = 1$. Solving for the coefficient yields

$$\gamma(t) = \frac{\mathbb{E}\,\Phi_{xxx}^2}{2\mathbb{E}a^3}. \tag{8}$$

At first (8) holds almost everywhere. Its denominator is at least two, because $a \ge 0$ and $\mathbb{E}a^3 \ge (\mathbb{E}a^2)^{3/2} = 1$. The moments are continuous, so right continuity of γ extends (8) to every time and proves that γ is continuous. Evenness gives $\Phi_{xxx}(0,0) = 0$, hence $\gamma(0) = 0$.

Smoothness follows by induction, as in Chen (2026), Proposition 4.6 and Lemma 3.14. Fix $T < 1$ and consider the polynomial moments

$$M_P(t) = \mathbb{E}P(\Phi_x(t, X_t), \dots, \partial_x^m \Phi(t, X_t)),$$

where P is a polynomial in any finite number m of variables. Spatial differentiation of the PDE and Itô's product rule give polynomials Q_P, R_P, involving finitely many further spatial derivatives, such that

$$M_P(t) - M_P(0) = \int_0^t [M_{Q_P}(s) + \gamma(s) M_{R_P}(s)]\,ds.$$

The identity involves γ and none of its derivatives. Bounded spatial derivatives on the fixed strip make every moment continuous. If $\gamma \in C^k([0,T])$ and every M_P is C^k, the integral identity makes every M_P of class C^{k+1}, and then (8) makes γ of class C^{k+1}. The case $k = 0$ has just been proved, so $\gamma \in C^\infty([0,T])$ for every $T < 1$. The method of polynomial moments is that of Auffinger and Chen (2015), Lemma 2 and the proof of Theorem 2(ii), and the companion, §9.1, gives the polynomials explicitly.

The two differentiated moments are now

$$\frac{d}{dt}\mathbb{E}\,\Phi^2_{xxx} = \mathbb{E}[\Phi^2_{xxxx} - 6\gamma a\Phi^2_{xxx}], \qquad \frac{d}{dt}\mathbb{E}a^3 = \mathbb{E}[3a\Phi^2_{xxx} - 3\gamma a^4].$$

For a smooth order parameter, differentiating $q'' = \mathbb{E}\,\Phi^2_{xxx} - 2\gamma\mathbb{E}a^3$ therefore gives

$$q''' + 2\gamma'\,\mathbb{E}a^3 = \mathcal{C}_\gamma(t). \tag{9}$$

On a constant interval $\gamma' = 0$, and (9) is the second line of (4). For the minimiser $q = t$, so $q''' = 0$ and the left side is $2\gamma'\mathbb{E}a^3$. Strict increase and $\gamma(0) = 0$ give $\gamma(t) > 0$ for $t > 0$, so Proposition 5 applies and gives

$$2\gamma'(t)\,\mathbb{E}[a(t,X_t)^3] \geq \frac{\gamma(t)^2}{1000}\mathbb{E}[a(t,X_t)^4] > 0 \qquad (0 < t < 1). \tag{10}$$

Cauchy–Schwarz and $\mathbb{E}a^4 \geq (\mathbb{E}a^2)^2 = 1$ give

$$\mathbb{E}a^3 \leq \sqrt{\mathbb{E}a^2\,\mathbb{E}a^4} = \sqrt{\mathbb{E}a^4} \leq \mathbb{E}a^4,$$

so (10) implies $\gamma'(t) \geq \gamma(t)^2/2000$ for $0 < t < 1$. □

Remarks. (i) *The crossing property and positivity.* By (9), at the smooth minimiser both q'' and q''' vanish identically, and $\gamma'(t) > 0$ is equivalent to $\mathcal{C}_\gamma(t) > 0$. The crossing property does not give this. As stated, Chen (2026), Proposition 4.2 concerns the minimiser, and γ_δ is not one. Moreover $q''_\delta(t) = 0$ need not hold for any $\delta > 0$, and even if it did, the strict conclusion $\mathcal{C}_{\gamma_\delta}(t) > 0$ would yield only $\mathcal{C}_\gamma(t) \geq 0$ as $\delta \to 0$. Proposition 5 needs neither hypothesis, and its right side converges to $\gamma(t)^2\,\mathbb{E}a^4/1000 > 0$.

(ii) *The constant.* The constant $1/2000$ is not sharp. Lemma 3, Proposition 5 and (10) hold with the coefficient κ of (6) in place of $1/1000$, since each limit preserves a non-strict inequality and its constant. The argument above then gives $\gamma'(t) \geq \kappa\gamma(t)^2/2$, and $\kappa/2 = 1661507/3042000000 > 1/2000$. The case of §3.1 bounds what (6) can give, since there its best constant is $2/5$.

(iii) *Related arguments at a fixed time.* Lopatto's v4, Proposition 5.1 and Corollary 5.6 give qualitative crossing for arbitrary positive-temperature probability measures. Lopatto's external-field paper, Theorem 2.3 and §8 uses a spatial inequality at a fixed time for support and positive density at positive temperature, and its Appendix B gives the strict positivity of the density at zero field. Proposition 5 treats the terminal value $|x|$ with an explicit lower bound. The rescaling between the two settings is in the companion, §2.3.

5. Consequences for optimisation

The results of this section use Theorem 1 only. Strict positivity of the density is not needed.

Full support has the following interpretation for near-ground states. For each fixed $\rho \in [0,1)$ and $\delta, \eta > 0$, with probability tending to one there are configurations σ, τ with $|R(\sigma,\tau) - \rho| < \delta$ whose energies are both at least $n(P_* - \eta)$. This follows from Auffinger, Chen and Zeng (2020), Proposition 1, equation (5) in the Gaussian normalisation above. Strict increase of γ is also the no-overlap-gap assumption under which the algorithm of El Alaoui, Montanari and Sellke finds near-ground states (§5.2).

5.1 A martingale formula for the ground-state energy

Set $M_t = u(t, X_t)$, the bounded gradient martingale, which satisfies $dM_t = a(t, X_t)\, dB_t$. Its quadratic variations distinguish the two moments of a:

$$d\langle M \rangle_t = a(t, X_t)^2\, dt, \qquad d\langle M, B \rangle_t = a(t, X_t)\, dt.$$

Full support gives $\mathbb{E}a(t, X_t)^2 = 1$, so each increment $M_t - M_s$ has unconditional variance $t - s$. The first moment of a gives instead the covariance with the driving noise. Let U be the almost-sure and L^2 limit of M_t. The value identity is

$$P_* = \mathbb{E}[UB_1] = \int_0^1 \mathbb{E}[a(t, X_t)]\, dt. \tag{11}$$

Boundedness gives $|U| \le 1$, while $\mathbb{E}M_t^2 \to 1$ gives $\mathbb{E}U^2 = 1$, so $U^2 = 1$ almost surely.

The cancellation behind (11) runs as follows. Terminal convergence of the convex value functions gives $UX_1 = |X_1|$ (the equality is automatic on $X_1 = 0$), and Itô's formula gives $\mathcal{P}(\gamma) = \mathbb{E}|X_1| - \int_0^1 t\gamma(t)\, dt$. Multiplying $X_1 = B_1 + \int_0^1 \gamma(t) M_t\, dt$ by U cancels that integral, because $\mathbb{E}[UM_t] = t$, and the Parisi formula then gives (11). OpenAI (2026a), Corollary 7.1 and the companion, §6, prove the endpoint limits and the stochastic-integral covariance. Moreover, convexity and Cauchy–Schwarz give $0 \le \mathbb{E}a \le 1$, so

$$0 \le P_* - \int_0^T \mathbb{E}[a(t, X_t)]\, dt \le 1 - T.$$

The gradient-martingale and value methods have antecedents in Montanari, preprint v2, Lemmas 3.2–3.4, El Alaoui, Montanari and Sellke, preprint v1, Lemmas 6.5 and 6.13 and Sellke, preprint v4, Proposition 3.10 and Lemma 3.11, p. 17.

5.2 The classical algorithm

Theorem 1 gives a strictly increasing minimiser in $\mathcal{U}$, which is the no-overlap-gap condition, Assumption 2 of El Alaoui, Montanari and Sellke, preprint v1. Their Corollary 2.2 then gives, for every fixed $\varepsilon > 0$, a randomised algorithm that returns a configuration within εn of the finite SK maximum with probability tending to one over couplings and algorithmic randomness. It costs $C(\varepsilon)n^2$ real-arithmetic operations when the prescribed scalar functions and normalisers are available. This count includes no separate bound for numerical preprocessing or finite precision. OpenAI (2026a), Corollary 7.3 and the companion, §9.4, work out the Gaussian convention and the operation count. Chen already notes this consequence in §1, p. 4. In the notation of El Alaoui, Montanari and Sellke the martingale multiplier is a and the drift is γu, and $\mathbb{E}a^2 = 1$ is the variance normalisation that their incremental approximate message passing requires.

5.3 A fixed QAOA circuit before the thermodynamic limit

Let $\widehat{Z}_i, \widehat{X}_i$ be Pauli operators, and define the cost and mixer by

$$C_{n,J} = \frac{1}{\sqrt{n}} \sum_{i<j} J_{ij} \widehat{Z}_i \widehat{Z}_j, \qquad B_n = \sum_i \widehat{X}_i.$$

The eigenvalues of the cost are $H_n(\sigma)$. QAOA alternates cost evolution and mixing, starting from the uniform state $|+\rangle^{\otimes n}$ (Farhi, Goldstone and Gutmann (2014)). For a fixed sequence of angles $W = (\theta_1, \beta_1, \dots, \theta_p, \beta_p)$, write

$$|\psi_{n,J}(W)\rangle = e^{-i\beta_p B_n} e^{-i\theta_p C_{n,J}} \cdots e^{-i\beta_1 B_n} e^{-i\theta_1 C_{n,J}} |+\rangle^{\otimes n},$$

$$v(W) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_J \langle \psi_{n,J}(W) | C_{n,J} | \psi_{n,J}(W) \rangle.$$

Farhi, Goldstone, Gutmann and Zhou (2022) established this limit for fixed angles. OpenAI (2026b), Theorem 1.1 proves that for every $\varepsilon > 0$ there is one finite W with $v(W) \geq P_* - \varepsilon$, whose depth and angles depend on ε but not on n or J. This was conjectured in Basso, Farhi, Marwaha, Villalonga and Zhou (2022), §7, equation (7.2).

The proof uses Theorem 1 only through the following finite quantity. Fix the minimiser and its diffusion, a cutoff $0 < T < 1$, and a mesh $t_j = j\Delta$ with $\Delta = T/N$. From independent standard normals $Z_1, \dots, Z_{N+1}$, form the Euler scheme and the normalised increments

$$\begin{aligned} Y_0 &= 0, \\ Y_j &= Y_{j-1} + \sqrt{\Delta} Z_j + \Delta \gamma(t_{j-1}) u(t_{j-1}, Y_{j-1}), \\ h_j &= \frac{a(t_{j-1}, Y_{j-1})}{\sqrt{\mathbb{E} a(t_{j-1}, Y_{j-1})^2}}, \qquad g_j = h_j Z_j \quad (1 \leq j \leq N), \end{aligned}$$

and the randomly rounded spin $F_N = \operatorname{sgn}\{u(T, Y_N) + 2\Phi_{\mathrm{G}}(Z_{N+1}) - 1\}$, where Φ_{G} is the standard normal distribution function, so that $\mathbb{E}[F_N \mid Z_1, \dots, Z_N] = u(T, Y_N)$. OpenAI (2026a), Proposition 7.2 proves that the denominators are positive for all sufficiently large N and that

$$S_{T,N} := \sum_{j=1}^{N} \mathbb{E}[F_N g_j]\, \mathbb{E}[F_N Z_{j+1}] \;\longrightarrow\; I_T := \int_0^T \mathbb{E}[a(t, X_t)]\, dt \qquad (N \to \infty). \tag{12}$$

Since h_j depends only on $Z_1, \dots, Z_{j-1}$, the g_j are centred and orthonormal in L^2. The denominators are the counterparts in the Euler scheme of $\mathbb{E} a(t, X_t)^2 = 1$. The proof of (12) is reconstructed in the companion, §7.

Proposition 6 (circuit approximation; a consequence of OpenAI (2026b), §§2–5). For each fixed T, N in the preceding construction with finite positive normalisers, and every $\eta > 0$, there is a finite sequence of angles W, independent of n and J, such that

$$|v(W) - S_{T,N}| < \eta.$$

We do not reproduce its proof, which occupies the companion, §§8.1–8.6. Near-optimality follows from the decomposition

$$P_* - v(W) \leq (P_* - I_T) + |I_T - S_{T,N}| + |S_{T,N} - v(W)|. \tag{13}$$

Choose T so that $1-T<\varepsilon/3$, then N so that the second term is below $\varepsilon/3$, and finally apply Proposition 6 with $\eta<\varepsilon/3$. These finite choices give $v(W)>P_*-\varepsilon$ before n tends to infinity. The construction gives no bound on the depth or on the magnitudes of the angles in terms of the accuracy, and no efficient procedure for finding the angles.

Corollary 7 (a measured configuration on the same instance). For every $\varepsilon>0$ there is a finite sequence of angles W_ε, chosen independently of n and J, such that one computational-basis measurement Σ satisfies

$$\Pr_{J,\Sigma}\Big\{\max_\sigma H_{n,J}(\sigma)-H_{n,J}(\Sigma)\le\varepsilon n\Big\}\longrightarrow 1\qquad(n\to\infty).\tag{14}$$

Proof. For a fixed W, put $Y_n=H_{n,J}(\Sigma)/n$ and $K_n=\max_\sigma H_{n,J}(\sigma)/n$. Farhi, Goldstone, Gutmann and Zhou, §4, equations (13)–(22) prove

$$\mathbb{E}_{J,\Sigma}Y_n\to v(W),\qquad \mathbb{E}_{J,\Sigma}Y_n^2\to v(W)^2.$$

The second moment is $\mathbb{E}_J\langle(C_{n,J}/n)^2\rangle$, so it includes measurement variance as well as disorder fluctuations. Thus $Y_n\to v(W)$ in joint L^2. Gaussian concentration of the maximum and its defining mean limit give $K_n\to P_*$ in probability. Choose W with $P_*-v(W)<\varepsilon/3$. With probability tending to one, both $K_n-P_*<\varepsilon/3$ and $v(W)-Y_n<\varepsilon/3$, which proves (14). The two energies use the same J, and the argument needs no independence between them. □

Formal verification

The proofs above do not rely on the formalisation. OpenAI's formalisation in Lean with mathlib contains two statements that bear on them. The formal statement of full support asserts the support and moment conclusions of Theorem 1, with an admissible minimiser as its hypothesis, and constructs the associated diffusion. Its proof includes the rational identity (5) and the integrated inequality (6). A separate formal statement of the scalar consequences covers the value identity (11) and the finite Gaussian limit (12), again with a minimiser as hypothesis, and has its own proof.

Both statements are published as challenges for Comparator, a tool that checks that a Lean proof establishes a stated theorem. Two independent reruns report that Comparator accepts OpenAI's proofs of both: MathVet, on 8 October 2026, and C. Toledo, on 10 October 2026, the latter also with a second proof checker. We rebuilt the proof of full support only (companion, §13).

The formalisation has three limitations. Neither formal statement covers the strict positivity of the density, the inequality at a fixed time or the convergence of QAOA. The two statements use separate models of the order parameter, which have not been related to each other, and the companion compares each with the printed statement in §12. Finally, in the reading edition of this paper each passage is linked to Lean declarations. These links were made by AI agents and have not been checked by a mathematician, and they do not certify that a passage and a declaration agree.

The identity (5), with Table 1, and the numerical values of §3 are reproduced by two short scripts, one for the identity in exact rational arithmetic and one for the solvable case.

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