FINITE-TIME BLOW-UP FOR THE NAVIER–STOKES EQUATIONS

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Abstract. Suppose the unforced three-dimensional Navier–Stokes equations have a smooth nonzero whole-space solution u^* before time T that repeats on shrinking scales up to an orthogonal transformation and satisfies $|u^*(t,x)| \le C/(|x| + \sqrt{T-t})$. From this singular flow we construct finite-energy solutions with smooth divergence-free data, compactly supported on $\mathbb{R}^3$ or periodic on $\mathbb{T}^3$. Their rescaled velocity gradients converge to the profile gradient in L^2, globally on $\mathbb{R}^3$ and on expanding balls in similarity coordinates on the torus. The whole-space gradient norm in L^2 and the torus H^1 norm grow between positive multiples of the inverse fourth root of the time remaining. We cut off the whole-space datum and adjust it in finitely many directions, without a spectral stability assumption. A second theorem constructs the required profile from a family of approximations satisfying explicit inequalities. Alpöge and Buckmaster report verifying sufficient finite-data conditions by interval arithmetic. Their cited release contains the conditional proofs and omits the data and interval calculation (Subsection 1.5).

Contents

1. Introduction 2
2. Reduction of Theorem 1.1 11
3. Preliminaries 13
4. The equation linearized about u^* 18
5. Blow-up on $\mathbb{R}^3$ 25
6. Blow-up on the torus 26
7. From a certificate to a profile 32
Appendix A. Proofs of the lemmas of Sections 2 and 3 51
Appendix B. From represented data to the analytic certificate 53
Appendix C. The computation of the approximate profile 78
Acknowledgments 84
References 85

1. Introduction

Can a solution whose initial velocity vanishes outside a ball develop the same singularity as a flow with a slowly decaying tail? Theorem 1.1 gives such data for every nonzero recurrent flow satisfying the decay bound below. We study the incompressible Navier–Stokes equations in $\mathbb{R}^3$ without external force,

$$\partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p, \qquad \operatorname{div} u = 0. \tag{1.1}$$

Here u is velocity, p is pressure and the viscosity is one. For divergence-free data, Fefferman asks whether smooth, rapidly decaying initial velocities on $\mathbb{R}^3$ always give global smooth solutions of bounded energy, and whether smooth periodic data always give global smooth periodic solutions on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ [53, pp. 1–2].

In Leray's self-similar ansatz [70, §20, pp. 224–225], a flow approaching a singular time T has length scale $\sqrt{T-t}$ and velocity scale $(T-t)^{-1/2}$. In these units the velocity is independent of time. Here the rescaled velocity can evolve and return to its starting field after a fixed interval of logarithmic time. After each return the flow repeats the motion on a smaller length scale and at greater speed. The field may also turn or reflect at each return.

A nonzero flow that repeats this way already blows up at T, but its initial velocity need not have finite energy. In fact every time slice has infinite energy when the profile returns to the same field ($R = I$), as Barker and Prange's estimate shows in Subsection 1.3. There is nevertheless a norm in which the distant tail is small. Since the initial velocity gradient is $O(|x|^{-2})$ by Remark 1.2, its squared L^2 norm outside radius r is $O(\int_r^\infty \rho^{-4}\rho^2\, d\rho) = O(r^{-1})$. Cutting off a vector potential removes the tail at this cost in squared gradient norm and preserves zero divergence. A small initial error can still grow through successive returns. The proof adjusts the cutoff datum in finitely many directions to obtain the required decay of the error.

From a profile to finite-energy data. On $\mathbb{R}^3$, (1.6) compares the rescaled gradients on the entire space. On $\mathbb{T}^3$, (1.7) makes the comparison on a ball of fixed physical radius, which expands in similarity coordinates. Here one must cut off and periodize the evolving flow. The cutoff introduces an error in the equation throughout the motion. Section 6 removes that error and constructs an unforced periodic solution.

From an approximate family to a profile. A stationary profile with this decay is excluded by Tsai's theorem [86, Theorems 1–2, p. 31]. Theorem 7.1 instead seeks a time-periodic flow. A family of approximate profiles, together with data satisfying the certificate conditions, yields the required exact profile. Corollary 7.2 then gives breakdown on both domains in Fefferman's formulation [53, (A)–(D), pp. 1–2], as detailed in Subsection 2.3.

1.1. A recurrent profile and its localization.

Fix the time $T > 0$ at which the scales shrink to zero. Write $y = x/\sqrt{T-t}$ for position on the shrinking length scale and $s = \log(T/(T-t))$ for logarithmic time. Time $t = 0$ corresponds to $s = 0$, and $t \uparrow T$ to $s \to \infty$. In these variables W denotes the rescaled velocity and q the rescaled pressure. Let $S > 0$, and let $R \in O(3)$ be a linear isometry of $\mathbb{R}^3$. A *profile* with period S and isometry R is a pair (W, q) of smooth functions $W : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, not identically zero, and $q : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that, on $\mathbb{R} \times \mathbb{R}^3$,

$$\begin{gathered} \partial_s W + \tfrac{1}{2}W + \tfrac{1}{2}(y \cdot \nabla)W - \Delta W + (W \cdot \nabla)W + \nabla q = 0, \qquad \operatorname{div} W = 0, \\ W(s+S, y) = R\, W(s, R^{-1}y), \end{gathered} \tag{1.2}$$

and, for a constant C_W and all $(s, y) \in \mathbb{R} \times \mathbb{R}^3$,

$$|W(s,y)| \le C_W(1+|y|)^{-1}, \qquad |\nabla W(s,y)| \le C_W(1+|y|)^{-2}. \tag{1.3}$$

In the original variables the velocity is

$$u_T^*(t,x) = (T-t)^{-1/2}\, W\!\left(\log\frac{T}{T-t}, \frac{x}{\sqrt{T-t}}\right) \qquad (t<T,\ x\in\mathbb{R}^3), \tag{1.4}$$

which, with pressure $(T-t)^{-1}q(s,y)$, solves (1.1) on $(-\infty,T)\times\mathbb{R}^3$ by the first two equations of (1.2). Choose $s_0\in[0,S)$ and y_0 with $W(s_0,y_0)\neq 0$, which is possible by recurrence. For $n=0,1,\dots$ set $t_n = T(1-e^{-s_0-nS})$ and $\ell_n=\sqrt{T-t_n}$. Iterating (1.2) gives

$$u_T^*(t_n, \ell_n R^n y_0) = \ell_n^{-1} R^n W(s_0,y_0).$$

From one return to the next, the remaining time shrinks by e^{-S}, lengths shrink by $e^{-S/2}$ and speeds grow by $e^{S/2}$. Thus the velocity diverges in magnitude along a sequence tending to $(T,0)$. The last equation of (1.2) gives the scaling law at every time, with $\lambda = e^{-S}$,

$$u_T^*(T-\lambda(T-t),x) = \lambda^{-1/2} R\, u_T^*(t, \lambda^{-1/2}R^{-1}x) \qquad (t<T,\ x\in\mathbb{R}^3). \tag{1.5}$$

In the variables (t,x), which we call Eulerian, the bounds (1.3) read

$$|u_T^*(t,x)| \le \frac{C_W}{|x|+\sqrt{T-t}}, \qquad |\nabla u_T^*(t,x)| \le \frac{C_W}{\bigl(|x|+\sqrt{T-t}\bigr)^2}.$$

Bradshaw and Tsai call u_T^* backward discretely self-similar when $R=I$ and backward rotated discretely self-similar when R is a rotation [13, (1.5)–(1.9), preprint pp. 2–3]. If R reverses orientation, two periods give the rotation R^2. The velocity bound is a Type I bound at $(T,0)$. Remark 1.2 derives the gradient bound from it.

A *smooth solution* on $[0,T)\times\mathbb{R}^3$, $0<T\le\infty$, consists of a velocity and pressure smooth up to $t=0$ satisfying (1.1) and $u(0,\cdot)=u_0$. On a closed time interval we require smoothness at both endpoints. We write $u(t)=u(t,\cdot)$, and we say that u has *finite energy* if $\|u(t)\|_{L^2}$ is bounded in t. We identify functions on $\mathbb{T}^3$ with functions on $\mathbb{R}^3$ of period one in each coordinate. A smooth solution on the torus has both velocity and pressure periodic in this sense. Write B_r for the open ball about the origin of radius r, and use the $H^1([0,1]^3)$ norm on the torus.

Theorem 1.1. *Let $S>0$, and let R be a linear isometry of $\mathbb{R}^3$, that is, a rotation or a rotation composed with a reflection. Suppose that there exists a profile (W,q) with period S and isometry R, and let u_T^* be the field (1.4). Then there exist a divergence-free $u_0\in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and a smooth solution u of (1.1) on $[0,1)\times\mathbb{R}^3$ with datum u_0 and finite energy which blows up at time 1: there exist $\kappa\in(0,\frac14)$ and constants $0<c\le C$ such that*

$$\begin{aligned} c\,(1-t)^{-\frac14} \le \|\nabla u(t)\|_{L^2} \le C\,(1-t)^{-\frac14}, \\ \|\nabla(u(t)-u_1^*(t))\|_{L^2} \le C\,(1-t)^{\kappa-\frac14} \end{aligned} \qquad (0\le t<1). \tag{1.6}$$

Moreover, there exist $T\in(0,1]$, a divergence-free datum in $C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ and a smooth solution u_{per} of (1.1) on $[0,T)\times\mathbb{T}^3$ with this datum which blows up at time T at the same rate:

$$\begin{aligned} c\,(T-t)^{-\frac14} \le \|u_{\mathrm{per}}(t)\|_{H^1(\mathbb{T}^3)} \le C\,(T-t)^{-\frac14}, \\ \|u_{\mathrm{per}}(t)-u_T^*(t)\|_{H^1(B_{1/16})} \le C\,(T-t)^{\kappa-\frac14} \end{aligned} \qquad (0\le t<T). \tag{1.7}$$

Following the profile. Write $u(t,x)=(1-t)^{-1/2}U(s,y)$ in the whole space. Then (1.6) gives $\|\nabla_y(U(s)-W(s))\|_{L^2(\mathbb{R}^3)}\le Ce^{-\kappa s}$. The velocity error obeys the same bound in L^6 by (3.1).

In physical coordinates the absolute gradient error can grow. Its relative size satisfies

$$\frac{\|\nabla(u(t)-u_1^*(t))\|_{L^2(\mathbb{R}^3)}}{\|\nabla u_1^*(t)\|_{L^2(\mathbb{R}^3)}} = O((1-t)^\kappa).$$

On the torus write $\delta = T - t$ and $U_{\mathrm{per}}(s, y) = \sqrt{\delta}\, u_{\mathrm{per}}(t, \sqrt{\delta}\, y)$ using the periodic lift. The second line of (1.7) gives $\|\nabla_y(U_{\mathrm{per}}(s) - W(s))\|_{L^2(B_{1/(16\sqrt{\delta})})} \le C\delta^\kappa$. This comparison holds in a ball of fixed physical radius $1/16$, which expands in similarity coordinates.

The blow-up rate. The change of variables gives $\|\nabla u_T^*(t)\|_{L^2} = (T-t)^{-1/4}\|\nabla W(s)\|_{L^2}$. The function $s \mapsto \|\nabla W(s)\|_{L^2}$ is continuous, positive and S-periodic (Lemma 4.2). In the whole space, (1.6) and the reverse triangle inequality therefore give $(1-t)^{1/4}\|\nabla u(t)\|_2 = \|\nabla W(-\log(1-t))\|_2 + O((1-t)^\kappa)$. The leading coefficient is thus a positive periodic function of phase, possibly constant. For these divergence-free fields, integration by parts identifies the L^2 norm of the gradient with that of the vorticity, the curl of the velocity. The power $-1/4$ is Leray's lower-bound exponent [70, §21, p. 227]. The squared gradient norm is time integrable at blow-up, since $\int_t^1 (1-\tau)^{-1/2}\, d\tau = 2\sqrt{1-t}$. Hence the divergence of the gradient norm is compatible with finite total viscous dissipation in the energy identity (Appendix A.2).

Choice of initial data. There exist $\varepsilon_0 > 0$ and a fixed finite-dimensional space E of smooth, compactly supported, divergence-free fields with the following property. Given any divergence-free $w_0 \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\|\nabla(w_0 - W(0,\cdot))\|_{L^2} \le \varepsilon_0$, one can choose $\gamma \in E$ so that the datum $w_0 + \gamma$ gives a smooth finite-energy solution on $[0,1)$ satisfying (1.6) (Proposition 2.1 and Lemma 2.3). The datum u_0 in the theorem is one such sum. There is no symmetry requirement on w_0. For a fixed linear right inverse, the choice $w_0 \mapsto \gamma$ in Subsection 5.2 is Lipschitz on the full $\dot{H}^1_{\mathrm{div}}$ ball about $W(0)$ and gives zero at its centre. Restriction to compactly supported smooth w_0 gives the finite-energy data above. This freedom to prescribe w_0 and adjust only $\gamma \in E$ is the stability in finite codimension asserted here.

Remark 1.2. The bound on ∇W in (1.3) is a consequence of the bound on W and of the first two equations of (1.2), with a constant that depends only on C_W, and so is the bound $|\nabla^2 W(s,y)| \le C\,(1+|y|)^{-3}$. This follows, as in [79, Lemma 9.2 and Corollary 9.3], from the interior estimate for bounded smooth solutions of the Navier–Stokes equations [79, Lemma 9.1], which requires no bound on the pressure, applied, for $t < T$ and $x \in \mathbb{R}^3$, on the backward parabolic cylinder of radius $r = \frac{1}{2}\max\{|x|, \sqrt{T-t}\}$ about the point (t,x), on which the field (1.4) is bounded by C_W/r. Thus the hypothesis of Theorem 1.1 is that of a nonzero smooth solution of (1.2) with the first bound of (1.3).

To obtain the profile assumed in Theorem 1.1, we start from a family of approximate periodic flows, each with its own period. A certificate (Subsection 7.3) specifies the family, auxiliary fields and constants. Its inequalities bound the error in the equation and the size of a correction, with estimates for every admissible perturbation of each family member. They also ensure that the correction leaves a nonzero flow and that the auxiliary forces introduced to solve the equations can all be made to vanish. Theorem 7.1 selects one parameter and corrects the corresponding member to an exact profile with its period and prescribed symmetries. That profile repeats after a full period with $R = I$. Appendix B gives sufficient conditions on finitely many represented fields and numbers.

Figure 1 shows the computed field $\bar{W}$ from which the approximate profiles are formed. At the phase shown, its vorticity concentrates in a central column and in two layers, one on each side of $y_3 = 0$, with nineteen spiral arms in each. The computed period is $\bar{S} \approx 0.36$. Rotation through $2\pi/19$ about the column leaves the field unchanged. After half a period one recovers the same field rotated through $\pi/19$. These are symmetries of the approximate family. The

perturbations have the same symmetries. Hence the selected exact profile obeys the half-period rotation law with its own period (Remark 7.3). Its symmetry under a half-turn about a perpendicular axis also precludes a fixed shape rotating steadily about the column, as shown in Subsection 1.3.

The two spiral patterns turn about the column in opposite senses. Their motion suggests testing each group of harmonics in a frame turning with one of the patterns (Appendix B.2). A horizontal section in Figure 3 shows the nineteen arms. Subsection C.1 gives the vorticity balance that any exact periodic profile must satisfy over one period.

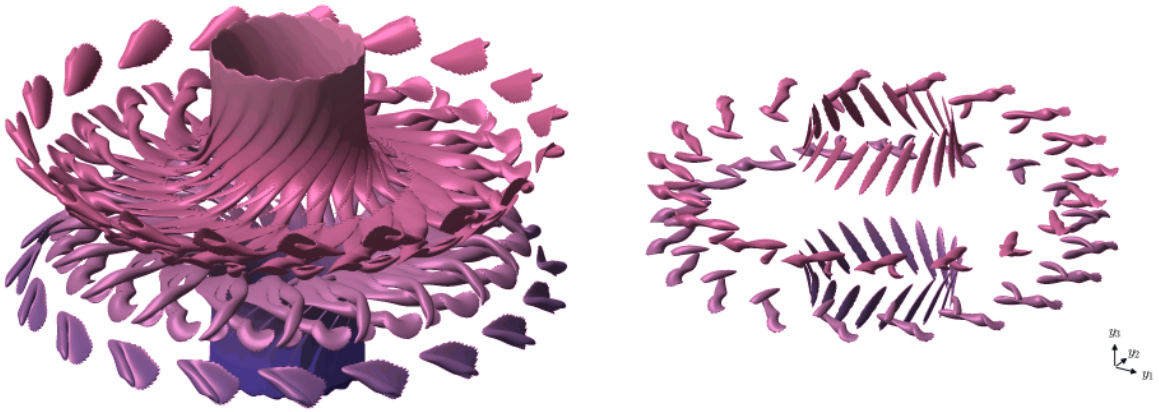


FIGURE 1. Vorticity of the computed field $\bar{W}$ at a single phase, $s = 0$. The two panels show the surfaces $|\operatorname{curl} \bar{W}| = 50$ (left) and $|\operatorname{curl} \bar{W}| = 85$ (right). These levels are about 0.30 and 0.51 of the maximum. Colour indicates height along the vertical column axis y_3. Figure 3 cuts horizontally through the upper layer.

1.2. Ideas of the proof. *Changing the tail.* The field $u^* = u_1^*$ associated with a profile is an exact solution on $[0, 1)$. The decay in (1.3) places $W(s)$ in L^6 and its gradient in L^2 uniformly in s. We work in $\dot{H}^1_{\mathrm{div}}(\mathbb{R}^3)$, the space of divergence-free L^6 fields with square-integrable gradient. Proposition 4.1 gives the integral equation there. Once a solution has compactly supported smooth data, local H^1 theory and the energy inequality give smoothness and finite energy (Lemma 2.3). On the torus the corresponding space consists of periodic fields of mean zero.

Selecting the initial perturbation. Write $u(t, x) = (1-t)^{-1/2} U(s, y)$ and $v = u - u^*$. We seek exponential convergence of $U(s)$ to $W(s)$ in the gradient norm. In computing $\|\nabla v(t)\|_2$ the velocity scale and the derivative supply $(1-t)^{-1}$, while the square root of the volume element contributes $(1-t)^{3/4}$. A perturbation of size $e^{-\kappa s}$ in similarity variables thus gives the physical gradient error in (1.6). Which initial perturbations have this decay?

Sample a scalar mode once per period. Suppose $x_{n+1} = a x_n + f_n$, $|f_n| \le C\Theta^n$ and $0 < \Theta < \min(1, |a|)$. Require $x_n = O(\Theta^n)$. Then $a^{-n} x_n \to 0$, and summing the recurrence backwards gives

$$x_0 = -\sum_{j \ge 0} a^{-j-1} f_j, \qquad x_n = -\sum_{j \ge n} a^{n-j-1} f_j, \qquad |x_n| \le \frac{C\Theta^n}{|a| - \Theta}.$$

Any other initial value leaves a nonzero multiple of a^n. Even when $|a| < 1$, this part decays too slowly. In the nonlinear equation the contraction determines the inhomogeneous term and the trajectory together.

To see which perturbations can have this decay, evolve the linearized equation about W through one similarity period and undo the isometry R. This gives the monodromy operator $\mathcal{M}$ on $\dot{H}^1_{\mathrm{div}}$. On dropping the terms containing W, the evolution from 0 to $s > 0$ consists of heat flow for time $1 - e^{-s}$ followed by the dilation $f \mapsto \sqrt{\mu}\, f(\sqrt{\mu}\,\cdot)$ with $\mu = e^{-s}$. Heat flow is contractive, and this dilation multiplies the gradient norm by $\mu^{1/4}$. After one period we undo the same isometry R as for $\mathcal{M}$. The resulting free return map has norm at most $e^{-S/4}$. The fourth root here and in (1.6) comes from measuring one gradient on the shrinking volume element. Lemma 4.4 proves that the difference between $\mathcal{M}$ and the free return map is compact. In the integral for that difference, approximate the background in $\dot{H}^1$ by a smooth field supported in a ball. At each positive heat time, local L^2 compactness on the ball and smoothing of the product give a compact operator (Lemma 3.2). The bound (3.7), of order $\tau^{-3/4}$ at heat time τ, is integrable at zero (Lemma 4.4). Outside the disk of radius $e^{-S/4}$ the spectrum therefore consists of isolated eigenvalues of finite multiplicity. Choose $\Theta \in (e^{-S/4}, 1)$ with no spectrum on its circle. Only finitely many eigenvalues lie outside that circle (Lemma 4.5).

Set $\Theta = e^{-\kappa S}$, so $0 < \kappa < 1/4$. Even an eigenvalue of modulus between Θ and 1 gives decay, but slower than the required Θ^n after n periods. We impose no spectral stability condition on the profile. For a perturbation to decay at the chosen rate its physical-variable norm must satisfy

$$\sup_{0 \le t < 1} (1-t)^{\frac{1}{4}-\kappa} \, \|v(t)\|_{\dot{H}^1} < \infty.$$

On the spectral subspace inside the circle, forward summation of the inhomogeneous terms over past periods gives the required decay. On the finite-dimensional exterior subspace, decay fixes the initial component through a sum over all future periods. The power bounds (4.11) give convergence of both sums. The projections of the smooth compactly supported solenoidal fields are dense in the exterior subspace and hence span it, since it is finite dimensional. Choose one such field over each basis vector. Their span is the correction space E (Subsection 4.4). Its dimension and the chosen κ depend on the profile and the spectral circle. The resulting right inverse (4.8) and a nonlinear contraction give, for each w_0 near $u^*(0) = W(0)$ in $\dot{H}^1_{\mathrm{div}}$, an element $\gamma \in E$ and a solution with datum $w_0 + \gamma$ satisfying the displayed bound for v. The center, blow-up time and phase of the profile stay fixed.

Localization in the two domains. We can now choose w_0 by cutting off the tail of $u^*(0)$. A direct cutoff of velocity need not preserve zero divergence. The cutoff is applied to a vector potential, so that the field remains divergence-free: for $0 \le t < 1$ and a radius $r > 0$ we let $\bar{u}(t) = \mathrm{curl}(\eta(\cdot/r)\, A(t))$, where $A(t)$ is the vector potential of $u^*(t)$ given in (5.1) and η is a smooth cutoff that equals 1 on $B_{1/2}$ and vanishes outside B_1. The field $\bar{u}(t)$ equals $u^*(t)$ on $B_{r/2}$ and vanishes outside B_r, and by (1.3) its distance from $u^*(t)$ in $\dot{H}^1$ is at most a constant multiple of $r^{-1/2}$, uniformly in t (Lemma 5.1). On $\mathbb{R}^3$ choose r so that $Cr^{-1/2} \le \varepsilon_0$, the permitted distance from $W(0)$ in Proposition 2.1, and take $w_0 = \bar{u}(0)$ (Subsection 5.2). Only the datum is cut off on $\mathbb{R}^3$. Lemma 4.8 then gives a solution from the adjusted datum which approaches the exact reference flow in similarity variables.

On $\mathbb{T}^3$ we use the same contraction, this time near an approximate solution. We must localize the entire reference field to make it periodic. We first work at blow-up time one on $\mathbb{T}^3_L = \mathbb{R}^3/L\mathbb{Z}^3$, with L large. At each time cut off the potential of u^* at radius $L/8$, take its curl $\bar{u}(t)$ and periodize it. The resulting field u^{a} equals u^* on $B_{L/16}$. To bound the interaction with translated copies, Lemma 6.1 uses the $|x|^{-4}$ decay of the kernel of the projected heat flow on divergences. The resulting residual has order $L^{-1/2}$ in the weighted gradient norm (Subsection 6.2).

There is a second error when transferring the linear inverse to the torus. Apply the whole-space inverse on the interaction between the periodic input and the cutoff field. This product is supported in one ball. Equation (6.12) uses the resulting perturbation inside the ball and periodizes the compactly supported initial correction. The reconstructed operator is a right inverse up to an error of norm $O(L^{-1/2})$ (Lemma 6.3). For large L the error norm is less than one. Inverting identity plus this error by a Neumann series gives an exact right inverse. The nonlinear contraction now removes the residual of u^{a}. Finally $u_{\mathrm{per}}(t,x) = L\,u(L^2t, Lx)$ takes the solution on $\mathbb{T}^3_L$ to the unit torus, with blow-up time $T = L^{-2}$.

Finding the recurrent flow. The correction in Sections 4–6 changes the initial velocity. In Section 7 the unknown is the periodic orbit itself, including its period S. We set $\theta = s/S$ and give all trial orbits period one in θ. A translate of an exact orbit is again an orbit, so differentiation with respect to the shift gives a null direction of the linearized equation. A phase condition fixes the time origin. To see the roles of the other constraints and forces, consider the linear map $L(x,y) = (x,0)$. The augmented equations $L(x,y) + (0,\beta) = (f_1, f_2)$ and $y = c$ determine $x = f_1$, $y = c$ and $\beta = f_2$. The constraint fixes the free coordinate, and the auxiliary force measures the mismatch in the second equation. If the data depend on a parameter, the original equation holds where that mismatch vanishes.

We introduce one linear constraint for each auxiliary force coefficient. For a trial parameter α and a trial field g, Proposition 7.8 solves the linear equations for a velocity correction v and force coefficients β. Proposition 7.10 updates g to $(v\cdot\nabla)v$, the quadratic term in the correction. Equation (7.30) proposes a parameter change by applying an invertible matrix to the total force coefficients $\beta^{\mathrm{a}}(\alpha) + \beta$. The update projects the proposed parameter onto the allowed set $\mathcal{K}$. This joint update needs only the linear solution at each trial. At a fixed point $g = (v\cdot\nabla)v$. The certificate estimates now put the proposed parameter in $\mathcal{K}$ before projection. Its change is therefore zero, and invertibility gives $\beta^{\mathrm{a}}(\alpha) + \beta = 0$. The corrected orbit solves the unforced equation. Proposition 7.11 gives smoothness. Continuity in L^6 follows from (3.1) and continuity of dilation. For this unrotated recurrence, Chae and Wolf [26, Theorem 1.1, preprint p. 2] give the velocity decay in (1.3), and Remark 1.2 gives the gradient bound.

1.3. Background. *Restrictions on recurrent profiles.* A time-independent W gives a self-similar solution in Leray's sense. Nečas, Růžička and Šverák [76, Theorem 1, p. 291] exclude nonzero stationary profiles in $L^3(\mathbb{R}^3)$. Tsai [86, Theorems 1–2, p. 31] excludes them in $L^p(\mathbb{R}^3)$ for $3 < p < \infty$, and under local energy estimates. By (1.3) every slice of W lies in L^6. Tsai's result with $p = 6$ therefore forces any nonzero profile considered here to depend on s. Bradshaw and Tsai [13, Open Problem 5.1, preprint p. 22] ask whether every backward discretely self-similar solution with the Type I bound, possibly up to rotation, stays bounded through T. By (1.5), boundedness through T would force every profile value to vanish.

For unrotated discrete self-similarity ($R = I$), Chae and Wolf [26, Theorems 1.1 and 1.3, preprint pp. 2–3] prove that a smooth backward solution continuous into $L^p(\mathbb{R}^3)$, $3 \le p < \infty$, satisfies the Type I bound. Conversely, given the bound with constant C_W, they exclude nonzero solutions whose expanding spatial scale $e^{S/2}$ is sufficiently close to one. Pineau and Vicol [79, Theorems 1.6–1.7, pp. 6–7] give a quantitative exclusion. For sufficiently small S it also holds for rotations through angles $|\varphi| \le cS$, with both thresholds depending on C_W. The computed period alone therefore cannot decide applicability. A profile rotating continuously at a constant angular speed in s would obey a rotated recurrence for every period. Bradshaw and Tsai record the question for such profiles as a problem of Perelman [13, Open Problem 5.2, preprint p. 22]. For such steadily rotating profiles, [79, Theorem 1.4, p. 4] excludes angular speeds of sufficiently small or sufficiently large magnitude at a fixed Type I bound. Forward self-similar and discretely self-similar flows exist [64, 87, 12, 27], also with rotation [13].

The spatial bound in (1.3) puts $u_T^*(t)$ in weak $L^3(\mathbb{R}^3)$ uniformly in t. For $R = I$, Barker and Prange [98, Corollary 1.1, p. 720] give the size of the cubic integral. Translate the singular time to zero and use their result with $p = 6$. Smoothness and (1.3) give continuity into L^6 by dominated convergence on compact time intervals. At physical radius $r\sqrt{T-t}$ the cubic velocity integral equals $\int_{|y|<r} |W(s,y)|^3\,dy$. It is bounded above and below by positive multiples of $\log r$ for large r, uniformly in s. Thus no slice belongs to L^3. Since each belongs to L^6, interpolation excludes finite energy at every phase.

For finite-energy solutions from smooth compactly supported data on $\mathbb{R}^3$, Escauriaza, Seregin and Šverák [52, Theorem 1.3, preprint p. 5] exclude a uniform L^3 bound up to blow-up. Seregin [100, Theorem 1.1] proves that the L^3 norm tends to infinity at the first singular time. For the whole-space solution in (1.6) with $R = I$, one can locate part of this growth. Put $s = -\log(1-t)$ and choose the similarity radius $r = e^{\kappa s}$. The L^6 error after (1.6) and Hölder's inequality give $\|U(s) - W(s)\|_{L^3(B_r)} \le Cr^{1/2}e^{-\kappa s} = Ce^{-\kappa s/2}$. The $L^3(B_r)$ norm of $W(s)$ is comparable to $(\log r)^{1/3}$. The triangle inequality and the scale invariance of the cubic integral therefore give

$$\int_{|x|<(1-t)^{1/2-\kappa}} |u(t,x)|^3\,dx \asymp \log\frac{1}{1-t} \qquad (t \uparrow 1).$$

The comparison constants depend on the reference flow and the chosen κ.

Several criteria exclude axisymmetric Type I singularities [28, 29, 69, 81]. In particular, (1.3) bounds u_T^* both by $C_W/\sqrt{T-t}$ and by C_W/r for cylindrical distance $r > 0$ from the axis. An axisymmetric u_T^* would remain bounded through T by Koch, Nadirashvili, Seregin and Šverák [69, Theorem 6.2, preprint p. 21], and (1.5) would then force it to vanish. Nineteenfold symmetry allows angular dependence. The symmetry imposed in (B.1) also rules out a fixed shape turning steadily about the column axis. Indeed, suppose $W(s)$ were obtained from $W(0)$ by such a rotation through ωs. The half-turn about the y_1 axis fixes both fields by (7.39) and conjugates an axial rotation to its inverse. Hence rotations of $W(0)$ through ωs and $-\omega s$ give the same field for every s. If $\omega \neq 0$, this makes $W(0)$ invariant under every axial rotation. In either case W would be stationary, which Tsai's L^6 exclusion forbids.

Albritton and Barker [1, Theorem 1.1 and (1.5), preprint p. 2] characterize the existence of a suitable weak solution with a Type I singular point by the existence of a nontrivial mild bounded ancient solution satisfying their bounds on rescaled energy, dissipation, velocity and pressure. They also show that a nontrivial mild ancient solution uniformly bounded in weak L^3 along times tending to $-\infty$ yields a singular weak Leray–Hopf solution with forcing [1, (4.2) and Remark 4.3, preprint pp. 10, 12]. The unforced solutions here come from localizing a recurrent profile.

Finite rotational symmetry imposes quite different restrictions in two and three dimensions. Elgindi and Jeong [49] proved global existence and uniqueness for two-dimensional Euler with bounded vorticity invariant under rotation by $2\pi/m$, $m \ge 3$, even without decay at infinity. In three dimensions their octahedrally symmetric construction on $\mathbb{R}^3$ gives finite-time blow-up from bounded, piecewise smooth vorticity [50].

Drivas and Elgindi [45, Problem 6] ask whether scale-invariant Euler solutions can become singular within an admissible symmetry class. The rotating surface quasi-geostrophic solutions in [25] have threefold symmetry.

Localization and selection of initial data. Jia and Šverák [65, Lemma 2.7, preprint pp. 10–11] use a compact perturbation of the free evolution for a forward self-similar background. The forward periodic construction of [17, §19, p. 74] gives the essential spectral radius bound $e^{-S/4}$.

The vector-potential cutoff in Section 5 also occurs in [17, §19, p. 75]. For forward Navier–Stokes flow, Jia and Šverák [65, §5, preprint pp. 20–21] cut off an infinite-energy datum with small error in the perturbation norm. Elgindi, Ghoul and Masmoudi [48, Subsection 2.6] do this for Euler.

The choice of initial data has precedents in compressible flow. Merle, Raphaël, Rodnianski and Szeftel [74, 75] constructed self-similar Euler profiles and smooth spherical data whose Euler and Navier–Stokes solutions implode near them. For three-dimensional isentropic Euler, Buckmaster, Cao-Labora and Gómez-Serrano [16] obtained smooth imploding profiles for all adiabatic exponents larger than one. At exponent 7/5 they proved Navier–Stokes implosion for a finite-codimensional set of smooth radial data with density bounded away from zero and constant at infinity. Cao-Labora, Gómez-Serrano, Shi and Staffilani [24] removed the symmetry assumption for both systems on $\mathbb{R}^3$ and on the torus.

For two-dimensional compressible Euler, Chen, Cialdea, Shkoller and Vicol [31] obtained vorticity blow-up with finitely many conditions on radial data and an open set of swirl velocities. Higher-dimensional and non-isentropic implosions appear in [30] and [36], respectively. Buckmaster and Iyer [19] obtained two-dimensional compressible Euler shocks approaching an unstable Burgers profile, with stability modulo two directions.

The number of corrections depends on the profile. For the $C^{1,\alpha}$ self-similar incompressible Euler blow-up of Elgindi [47], one condition on axisymmetric perturbations gives finite-energy blow-up [48, Theorem 2]. For the nonlinear heat equation, Bricmont and Kupiainen [15, Theorems 1–2] choose one parameter per unstable direction. Collot, Raphaël and Szeftel [38, Theorem 1.2] construct a finite-codimensional Lipschitz manifold of data near a cutoff profile for the energy-supercritical heat equation, up to scaling and translation. Li and Sinai [71] choose data in ten-parameter families for complex-valued Navier–Stokes blow-up.

For nonlinear wave equations, Donninger and Schörkhuber [43], following [42] on wave maps, used the Lyapunov–Perron method to correct the unstable time-translation direction and then absorbed the correction by moving the blow-up time. This is the method that we use, except that we keep the modification of the data. Glogić, Hilditch and Wallauch [57] proved finite-codimensional stability of discretely self-similar blow-up for a wave maps type equation.

Albritton, Brué and Colombo [3, Theorem 1.1, p. 2] extended the forced nonuniqueness construction of [2] to a torus. Tao [84, proof of Theorem 12.2, preprint pp. 75–76] cut off a whole-space solution and embedded it in a large torus, choosing a force to absorb the error. In Section 6 the error of the cutoff is removed by the contraction, and the equations on the torus hold without force.

For Navier–Stokes, forward instability can separate two solutions with the same datum. Jia and Šverák [65] gave a spectral condition on a forward self-similar profile sufficient for nonuniqueness of Leray–Hopf solutions. Following Vishik's work on two-dimensional Euler [91, 92, 4], Albritton, Brué and Colombo [2] proved forced Leray–Hopf nonuniqueness. In the unforced case Buckmaster, Coiculescu and Li [17] followed a growing Floquet mode of a forward discretely self-similar solution. Here we follow the stable directions toward the blow-up time, and the solution converges to the profile. Other mechanisms give nonunique finite-energy weak solutions by convex integration [20, 18, 21], or two global solutions on the torus, smooth at positive times, from one BMO^{-1} datum [37].

Forcing and blow-up. For Euler, Elgindi obtains finite-energy blow-up with a uniformly $C^{1,\alpha}$ force [47, Remark 1.4]. Córdoba and Martínez-Zoroa construct Euler blow-up with a uniformly $C^{1,1/2-\varepsilon}$ force [39] and porous-media blow-up with a spatially smooth source [40]. The incompressible porous media, Boussinesq and Euler constructions in [7, 5, 6], respectively,

use forces smooth in both space and time up to blow-up. The Euler construction of [8] uses smooth periodic data without force.

Computer-assisted construction of profiles. Computer-assisted proofs first reduce a functional estimate to quantitative conditions and then enclose the numerical errors in checking those conditions [58]. Chen and Hou [32, 33] used such a reduction for blow-up from smooth data in the two-dimensional Boussinesq and three-dimensional axisymmetric Euler equations with boundary. Earlier work with Huang treated the De Gregorio and Hou–Luo models [34, 35]. Other applications include the smooth imploding profiles of [16] and the one-dimensional kernel in Castro, Córdoba and Gómez-Serrano's construction of rotating surface quasi-geostrophic solutions [25].

The discussion of related constructions in Subsection 7.10 compares the method with periodic Navier–Stokes flows under steady forcing on the torus [90] and in a square [9], and with discrete self-similarity in the Einstein–scalar field system [80] and the defocusing nonlinear wave equation [22]. Other computations locate candidate flows. Neural networks have produced approximate self-similar Euler and model profiles [93, 94]. Guillod and Šverák [59] found a symmetry-breaking bifurcation for forward self-similar Navier–Stokes solutions with large axisymmetric data. Ionescu, Jia and Palasek [63] computed unstable forward profiles in the axisymmetric class without swirl. Hou [61] reported numerical evidence of possible singularity for smooth finite-energy axisymmetric data.

1.4. Organization and conventions. For the passage from a profile to finite-energy data, Sections 2–6 prove Theorem 1.1. Section 3 gives the heat estimates, Section 4 chooses the initial correction, and Sections 5–6 localize the profile on the two domains. Appendix A supplies the auxiliary proofs, including smoothness (Lemma 2.3). For the construction of the profile, Section 7 can be read after Section 3. Appendix B gives the conditions on the represented data, and Appendix C describes the numerical search and the fields in Figure 1.

We use viscosity one. For any $\nu > 0$ the change $(u, p) \mapsto (\nu\, u(\nu t, x), \nu^2 p(\nu t, x))$ gives the corresponding solution with viscosity ν. From Section 2 on, the solutions that we construct blow up at time 1, except for the solution on $\mathbb{T}^3$, which is a dilation of one of them. The letters s and y are reserved for the self-similar variables. We write $\bar{B}_r$ for the closure of the ball B_r. For a field f on $\mathbb{R}^3$ we write $|\nabla f|^2 = \sum_i |\partial_i f|^2$, and similarly for derivatives of higher order. The Fourier transform is $\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi}\, f(x)\, dx$, with inverse $\mathcal{F}^{-1}$, and the Fourier coefficients of a periodic function are $\hat{f}(k) = \int_{[0,1]^3} e^{-2\pi i k\cdot x}\, f(x)\, dx$, $k \in \mathbb{Z}^3$. The Leray projection $\mathbb{P}$ removes the component parallel to each nonzero Fourier frequency. It therefore annihilates gradients at those frequencies. The gradient of a periodic pressure has zero mean, so the projection annihilates it on the torus as well. In Fourier variables $\mathbb{P}$ multiplies $\hat{f}(\xi)$, and on the torus $\hat{f}(k)$, by the matrix $\mathbb{P}_\xi = I - |\xi|^{-2}\, \xi \otimes \xi$, where $\mathbb{P}_0 = I$, so that on the torus it keeps the mean.

1.5. Formal statements and numerical verification. *Global smooth solutions.* Alpöge and Buckmaster's formal development (revision `ad53419`) uses Lean 4 [41] and Mathlib [73]. Its profile hypothesis adds $|\nabla^2 W| + |\nabla q| \le C(1 + |y|)^{-3}$ to (1.3). Remark 1.2 supplies the velocity derivative bounds. The pressure-gradient decay is an additional hypothesis in this comparison. For every positive viscosity the authors obtain smooth, compactly supported, divergence-free data on $\mathbb{R}^3$ for which there is no global smooth solution of bounded energy. They also obtain smooth divergence-free periodic data for which there is no global smooth solution with periodic velocity. The data may depend on viscosity. The change of frame at the end of this subsection explains the periodic pressure convention.

Rates and convergence. Under the formal decay hypotheses, the formalization for this edition proves the two-sided rate and the gradient error (1.6) on $\mathbb{R}^3$. The written proof in Section 6 gives the periodic rate and the local error (1.7). Their formal counterpart remains an open target in this edition.

The recurrent profile. The authors' certificate theorem constructs a nonzero weak recurrent profile. The edition's formal certificate theorem retains the selected parameter, the smooth symmetric profile and the bounds (7.38). Its hypotheses require, among other things, joint C^1 regularity of the approximate velocity and a specified decaying weak second spatial derivative. The original statement checks and the edition checks at `4e96459` record the compiled results. For a certified water-wave computation formalized in Lean see [82].

The numerical instance. In the authors' formal proof, the full finite-data conditions described in Appendix B imply the two nonexistence statements. Alpöge and Buckmaster report checking those conditions for their data by 128-bit interval arithmetic using Arb [66, §§1–2, preprint pp. 1–2] through FLINT. Their source repository supplies the paper and Lean proofs. The certificate file and interval calculation are absent there. The original abstract identifies the intended file by its SHA-256 digest `3f7a335b84b6c7631742cacc5343dfcd11cf72f5221ff79b2b9268790b4b22c5`.

Periodic pressure. For a smooth solution with periodic velocity, (1.1) makes ∇p periodic. Each increment $p(t, x + e_j) - p(t, x) = b_j(t)$ is therefore independent of x. Choose $a'' = -b$ with $a(0) = a'(0) = 0$. For $u(t, x + a(t)) - a'(t)$ the transport term from differentiating the shift cancels the one from subtracting a'. The remaining acceleration $-a''$ is absorbed by the pressure $p(t, x + a(t)) + a''(t) \cdot x$. Its jth increment is $b_j + a_j'' = 0$. The datum is unchanged, and both velocity and pressure are periodic as in Fefferman's erratum [53, Errata, unnumbered PDF p. 6].

1.6. Original authors' statement on AI use. Alpöge and Buckmaster describe their use of Claude in the computations, the original formal proof and the successive drafts of the paper. Their first-person account appears in full after the Acknowledgments.

2. Reduction of Theorem 1.1

2.1. The space X and the notion of solution. In physical time the linear evolution is the ordinary heat flow, which gives an integral equation in a space containing the profile. Recurrence identifies successive time intervals by dilation and the isometry R (Subsection 4.3). Lemma 2.3 recovers smooth velocities and pressures for the compactly supported whole-space data and the smooth periodic data of Theorem 1.1. We write $C^\infty_{c,\mathrm{div}}$ for the space of smooth, compactly supported, divergence-free fields $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, and $X = \dot{H}^1_{\mathrm{div}}(\mathbb{R}^3)$ for the space of the divergence-free fields $u \in L^6(\mathbb{R}^3; \mathbb{R}^3)$ with $\nabla u \in L^2$, with the norm $\|u\|_{\dot{H}^1} = \|\nabla u\|_{L^2}$. For $t > 0$ and a field u in $L^p(\mathbb{R}^3; \mathbb{R}^3)$, $1 \le p \le \infty$, let

$$e^{t\Delta} u = (4\pi t)^{-3/2} e^{-|\cdot|^2/(4t)} * u, \tag{2.1}$$

and $e^{0\Delta} u = u$. For $\tau > 0$ and a field f that is the sum of a field in $L^1(\mathbb{R}^3; \mathbb{R}^3)$ and a field in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ let

$$e^{\tau\Delta} \mathbb{P} f = \mathcal{F}^{-1}\big[e^{-4\pi^2 \tau |\xi|^2}\, \mathbb{P}_\xi \hat{f}\big], \tag{2.2}$$

and for $u, v \in X$ let

$$K(\tau)(u, v) = -\tfrac{1}{2}\, e^{\tau\Delta} \mathbb{P}\big[(u \cdot \nabla) v + (v \cdot \nabla) u\big]. \tag{2.3}$$

In (2.3) the projection removes the pressure and the heat flow advances the resulting solenoidal field over the lag τ. Since $u \in L^6$ and $\nabla v \in L^2$, Hölder's inequality gives $f = (u \cdot \nabla)v \in L^{3/2}$. On $\{|f| > 1\}$ this field lies in L^1, and on $\{|f| \le 1\}$ it lies in L^2. Thus $f \in L^1 + L^2$, as required in (2.2), and (2.3) is defined. The map $K(\tau)$ is bilinear and symmetric, it maps $X \times X$ into X (Lemma 3.1), and $K(\tau)(u,u) = -e^{\tau\Delta}\mathbb{P}[(u \cdot \nabla)u]$.

Let $a < b$. For a continuous map $u : [a,b) \to X$, Duhamel's formula for (1.1) on $[a,b)$ is

$$u(t) = e^{(t-a)\Delta}u(a) + \int_a^t K(t-t')(u(t'),u(t'))\,dt' \qquad (a \le t < b), \tag{2.4}$$

A mild solution on $[a,b)$ is a continuous X-valued velocity satisfying (2.4). For each $t < b$, the X norm of u is bounded on $[a,t]$. By (3.7) the integrand is bounded in X by a constant times $(t-t')^{-3/4}$ and is continuous for $t' < t$ (Lemma 3.3), so the integral exists in X. On $[a,b]$ we require (2.4) also at b. From now on a solution with values in X is a mild solution, unless we say that it is smooth. The identity (2.4) is the integral formulation of the Navier–Stokes equations of Kato and Fujita [68, (III), p. 245]; for the equations in the whole space see [67, (INT) and (2.2), p. 473]. On the torus the space $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$ of the periodic divergence-free fields of mean zero, the bilinear term $K_{\mathbb{T}}$ and the mild solutions in this space are defined in the same way in Subsection 3.6.

2.2. Blow-up on $\mathbb{R}^3$ and on the torus. On $\mathbb{R}^3$ we prove the following proposition.

Proposition 2.1. *Let (W,q) be a profile, and let u_1^* be the field (1.4) with $T = 1$. Then there exist $\kappa \in (0, \frac{1}{4})$, constants $0 < c \le C$ and $\varepsilon_0 > 0$, and a finite-dimensional subspace E of $C^\infty_{c,\mathrm{div}}$ with the following property. For every $w_0 \in X$ with $\|w_0 - W(0,\cdot)\|_{\dot{H}^1} \le \varepsilon_0$ there exist $\gamma \in E$ and a solution u on $[0,1)$ with $u(0) = w_0 + \gamma$ such that, for $0 \le t < 1$,*

$$c\,(1-t)^{-\frac{1}{4}} \le \|u(t)\|_{\dot{H}^1} \le C\,(1-t)^{-\frac{1}{4}}, \qquad \|u(t) - u_1^*(t)\|_{\dot{H}^1} \le C\,(1-t)^{\kappa-\frac{1}{4}}. \tag{2.5}$$

Moreover, there is a field $w_0 \in C^\infty_{c,\mathrm{div}}$ with $\|w_0 - W(0,\cdot)\|_{\dot{H}^1} \le \varepsilon_0$.

Sections 4–5 choose one correction space E and one radius ε_0 for the whole ball in X. In particular, w_0 need not have any symmetry of the profile. Subsection 5.2 gives a Lipschitz choice of γ on this ball, vanishing at $W(0)$. Take w_0 to be the solenoidal cutoff of Lemma 5.1. Then both w_0 and its correction are smooth and compactly supported.

On the torus we prove the following proposition, at a small blow-up time T.

Proposition 2.2. *Let (W,q) be a profile. Then there exist $T \in (0,1]$, $\kappa \in (0, \frac{1}{4})$, constants $0 < c \le C$, a divergence-free $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ and a solution u in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$ on $[0,T)$ with $u(0) = u_0$ such that, for $0 \le t < T$,*

$$c\,(T-t)^{-\frac{1}{4}} \le \|u(t)\|_{H^1(\mathbb{T}^3)} \le C\,(T-t)^{-\frac{1}{4}}, \quad \|u(t) - u_T^*(t)\|_{H^1(B_{1/16})} \le C\,(T-t)^{\kappa-\frac{1}{4}}, \tag{2.6}$$

where u_T^* is the field (1.4).

The first estimate in (2.6) measures the norm over the whole torus. The second compares the periodic solution with a field on $\mathbb{R}^3$ only on $B_{1/16}$, where the cutoff field of Section 6 agrees with u_T^* after dilation. The datum has mean zero since the trajectory takes values in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$, the space used in Lemma 2.3.

2.3. Proof of Theorem 1.1. Proposition 2.1, for a field $w_0 \in C^\infty_{c,\mathrm{div}}$, and Proposition 2.2 give mild solutions with smooth data, and it remains to show that these are smooth solutions of (1.1). We say that a mild solution u is a smooth solution if the fields $u(t)$ are the values at time t of a smooth solution with datum $u(0)$.

Lemma 2.3. *Let $0 < T < \infty$, and let u be a mild solution on $[0,T)$ with $u(0) \in C^\infty_{c,\mathrm{div}}$. Then u is a smooth solution of (1.1) on $[0,T)\times\mathbb{R}^3$, and*

$$\|u(t)\|_{L^2} \le \|u(0)\|_{L^2} \qquad (0 \le t < T).$$

Moreover, every mild solution in $\dot H^1_{\mathrm{div}}(\mathbb{T}^3)$ on $[0,T)$ whose datum is smooth is a smooth solution of (1.1) on $[0,T)\times\mathbb{T}^3$.

Fix $0 < b < T$. Uniqueness identifies u with the velocity furnished by local H^1 theory for as long as the latter exists [84, Theorems 5.1 and 5.4]. In the whole space, the energy inequality and $\max_{[0,b]}\|u(t)\|_{\dot H^1}$ bound its H^1 norm. On the torus use mean zero and (3.16). Local continuation thus gives smoothness through b. Since $b < T$ was arbitrary, this proves the assertion on $[0,T)$ (Appendix A.2).

Proof of Theorem 1.1. Let (W,q) be a profile, and let κ, c, C, ε_0 and E be given by Proposition 2.1. By its last assertion there is a field $w_0 \in C^\infty_{c,\mathrm{div}}$ with $\|w_0 - W(0,\cdot)\|_{\dot H^1} \le \varepsilon_0$, and for this field the proposition gives $\gamma \in E$ and a solution u on $[0,1)$ with $u(0) = w_0 + \gamma$ which satisfies (2.5). The datum $u_0 = w_0 + \gamma$ belongs to $C^\infty_{c,\mathrm{div}}$, because $E \subset C^\infty_{c,\mathrm{div}}$. By Lemma 2.3, u is a smooth solution of (1.1) on $[0,1)\times\mathbb{R}^3$ with datum u_0, and it has finite energy. Since $\|u(t)\|_{\dot H^1} = \|\nabla u(t)\|_{L^2}$, the bounds (2.5) are (1.6).

On the torus, let T and the solution on $[0,T)$ be given by Proposition 2.2; we denote this solution by u_{per}. Its datum is smooth, so that, by Lemma 2.3, u_{per} is a smooth solution of (1.1) on $[0,T)\times\mathbb{T}^3$, and (2.6) is (1.7).

Finally, since $1 - t \le 1$ and $T - t \le 1$, the bounds (1.6) and (1.7) hold with the smaller of the two exponents κ and of the two constants c and with the larger of the two constants C. □

The same argument, for an arbitrary field $w_0 \in C^\infty_{c,\mathrm{div}}$ with $\|w_0 - W(0,\cdot)\|_{\dot H^1} \le \varepsilon_0$, gives the stability in finite codimension stated after Theorem 1.1.

To see the consequence for Fefferman's formulation, suppose the compactly supported datum above had a smooth finite-energy solution through time 1. On the closed interval $[0,1]$ this solution has bounded H^1 norm by Tao [84, Corollary 11.1, preprint p. 68], since its initial velocity is in H^1 and the force is zero. Its velocity is that of an H^1 mild solution [84, Corollary 4.3, preprint p. 28]. Uniqueness [84, Theorem 5.4, preprint pp. 33–34] identifies it with u for $t < 1$, contrary to (1.6). On the torus, uniqueness for smooth periodic solutions with periodic pressure [84, Theorem 5.1, preprint pp. 28–29] gives the same contradiction to (1.7).

3. Preliminaries

Three properties of the heat evolution enter the proof. The factor $\tau^{-3/4}$ in (3.7) is integrable at zero, so the quadratic term can be integrated up to the current time. At each positive lag the same operator is compact when one velocity is fixed (Lemma 3.2), which controls the part of the period map outside the free spectral disk in Section 4. On $\mathbb{T}^3_L$ the heat and Sobolev estimates retain constants independent of L, as required for the passage to large period in Section 6. Appendix A.1 proves Lemmas 3.3–3.7. We write $\lesssim$ for an inequality up to a constant, absolute in this section and in Appendix A unless otherwise specified.

3.1. The space X. The fields $u \in L^6(\mathbb{R}^3;\mathbb{R}^3)$ with $\nabla u \in L^2$ form the completion of $C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)$ in the norm $\|\nabla u\|_{L^2}$, and the Sobolev inequality

$$\|u\|_{L^6} \lesssim \|\nabla u\|_{L^2} \tag{3.1}$$

follows componentwise from the scalar inequality [56, (II.3.11), p. 56]. To approximate a field in L^6 with gradient in L^2, cut it off at radius R and then mollify. For a cutoff $\chi_R(x) = \chi(x/R)$

equal to one on B_R and zero outside B_{2R}, the extra gradient term is bounded in L^2 by $\|u\|_{L^6(B_{2R}\setminus B_R)}\|\nabla\chi_R\|_3$, which tends to zero since the second factor is independent of R. This scalar cutoff gives density in the ambient space. Divergence zero defines a closed subspace, so X is a real Hilbert space. On $\mathbb{R}^3$, every field in X can be approximated in the gradient norm by smooth compactly supported divergence-free fields [60, Theorem 8, p. 89; definitions on pp. 64–65]. This last density statement is what allows the spectral correction in Subsection 4.4 to have compact support.

For every $u \in X$ the Fourier transform $\hat{u}$ is a locally integrable function, and by Plancherel's identity, for $j = 1, 2, 3$ and almost every ξ,

$$\widehat{\partial_j u}(\xi) = 2\pi i \xi_j \, \hat{u}(\xi), \quad \xi \cdot \hat{u}(\xi) = 0, \quad \|u\|_{\dot{H}^1}^2 = \int 4\pi^2 |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi. \tag{3.2}$$

For smooth approximants the last integral controls $|\xi|\hat{u}$ in L^2. Cauchy–Schwarz against $|\xi|^{-1} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ gives convergence in L^1 on each frequency ball. Convergence of the fields in L^6 identifies that limit with the distributional Fourier transform of u. The heat flow (2.1) maps X into itself, and on X it is the Fourier multiplier $e^{-4\pi^2 t|\xi|^2}$: for $u \in X$ and $t, t' \geq 0$,

$$\widehat{e^{t\Delta}u} = e^{-4\pi^2 t|\xi|^2}\hat{u}, \qquad e^{(t+t')\Delta}u = e^{t\Delta}e^{t'\Delta}u, \qquad \|e^{t\Delta}u\|_{\dot{H}^1} \leq \|u\|_{\dot{H}^1}, \tag{3.3}$$

and $t \mapsto e^{t\Delta}u$ is continuous in X on $[0, \infty)$, by dominated convergence in the weighted integral (3.2).

For $\mu > 0$, $R \in O(3)$ and any field u on $\mathbb{R}^3$ let

$$D_\mu u(x) = \sqrt{\mu}\, u(\sqrt{\mu}\, x), \qquad \rho_R u(x) = R\, u(R^{-1}x). \tag{3.4}$$

These are the dilations in the scaling of the Navier–Stokes equations and the action of the isometries; both map X into itself. A change of variables gives, for $u \in X$, $t \geq 0$ and $\mu, \mu' > 0$,

$$\|D_\mu u\|_{\dot{H}^1} = \mu^{1/4}\, \|u\|_{\dot{H}^1}, \qquad D_{\mu\mu'}u = D_\mu D_{\mu'}u, \qquad D_\mu e^{\mu t\Delta}u = e^{t\Delta}D_\mu u, \tag{3.5}$$

and $(\mu, u) \mapsto D_\mu u$ is continuous on $(0, \infty) \times X$: smooth approximation gives continuity for fixed u, and (3.5) controls a simultaneous change in u. For $R \in O(3)$, ρ_R is an onto isometry of X commuting with $e^{t\Delta}$ and D_μ.

3.2. The bilinear term.

Lemma 3.1. *There exists a constant C such that for all $\tau > 0$ and $f \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^3)$ the field $e^{\tau\Delta}\mathbb{P}f$ belongs to X and*

$$\|e^{\tau\Delta}\mathbb{P}f\|_{\dot{H}^1} \leq C\, \tau^{-3/4}\, \|f\|_{L^{3/2}}. \tag{3.6}$$

Moreover $e^{t\Delta}e^{\tau\Delta}\mathbb{P}f = e^{(t+\tau)\Delta}\mathbb{P}f$ for $t \geq 0$, and $\tau \mapsto e^{\tau\Delta}\mathbb{P}f$ is continuous from $(0, \infty)$ to X. In particular, there exists a constant C_K such that for all $\tau > 0$ and $u, v \in X$ the field $K(\tau)(u, v)$ belongs to X and

$$\|K(\tau)(u, v)\|_{\dot{H}^1} \leq C_K\, \tau^{-3/4}\, \|u\|_{\dot{H}^1}\, \|v\|_{\dot{H}^1}. \tag{3.7}$$

For (3.6), apply Kato's derivative estimate with $m = 3$, $p = 3/2$, $q = 2$, giving $(1 + 3/p - 3/q)/2 = 3/4$ [67, (2.3), (2.3′), p. 474]; $\mathbb{P}$ is bounded on $L^{3/2}$ [67, p. 473]. Hölder gives $\|(u \cdot \nabla)v\|_{3/2} \leq \|u\|_6 \|\nabla v\|_2$, so (3.1) gives (3.7). The semigroup identity follows from the multipliers. Near $\tau_0 > 0$, factor the evolution through time $\tau_0/2$ and use strong continuity on X.

The semigroup identity and the invariance of the Navier–Stokes equations under their scaling and under linear isometries give three identities for K: for all $u, v \in X$, $t \geq 0$, $\tau, \mu > 0$ and $R \in O(3)$,

$$\begin{aligned} e^{t\Delta} K(\tau)(u,v) = K(t+\tau)(u,v), \qquad \mu\, D_\mu K(\mu\tau)(u,v) = K(\tau)(D_\mu u, D_\mu v), \\ \rho_R K(\tau)(u,v) = K(\tau)(\rho_R u, \rho_R v). \end{aligned} \tag{3.8}$$

The last two follow by a change of variables in (2.2) and (2.3), since $\mathbb{P}_\xi$ is homogeneous of degree zero in ξ and $\mathbb{P}_{R\xi} R = R\, \mathbb{P}_\xi$.

For $\tau > 0$ the Oseen tensor O_τ has Fourier multiplier $e^{-4\pi^2\tau|\xi|^2}\mathbb{P}_\xi$. It is the velocity part of the fundamental solution of the non-stationary Stokes system [78, §5, (11), p. 41], [83, Russian original, §5, (40), p. 235]. Sections 6 and 7 use its spatial derivative, the integrable kernel of $e^{\tau\Delta}\mathbb{P}\,\mathrm{div}$. The tensor is smooth and for every integer $l \geq 0$ a constant C satisfies

$$|\nabla^l O_\tau(x)| \leq C\,(|x| + \sqrt{\tau})^{-3-l} \qquad (x \in \mathbb{R}^3,\ \tau > 0), \tag{3.9}$$

by Leray's estimate [70, §12, (2.10), p. 212]. Solonnikov proves the estimates in the Russian original [83, §5, (44), p. 237]. For the first spatial derivative the decay exponent is four. See also [87, (2.4), manuscript p. 5] and, for $l \leq 1$, [69, (3.6)–(3.7), manuscript p. 7]. Since $O_\tau(x) = \tau^{-3/2}\, O_1(x/\sqrt{\tau})$, we have $\|\nabla O_\tau\|_{L^p} = \tau^{-2+\frac{3}{2p}}\, \|\nabla O_1\|_{L^p}$ for $1 \leq p \leq \infty$, and these norms are finite by the bound for $l = 1$. For a field F on $\mathbb{R}^3$ with values in the 3×3 matrices we write $|F|^2 = \sum_{j,m} F_{jm}^2$ and

$$(\mathrm{div}\, F)_j = \sum_{m=1}^{3} \partial_m F_{jm}, \qquad (\nabla O_\tau * F)_i = \sum_{j,m=1}^{3} \partial_m (O_\tau)_{ij} * F_{jm},$$

and for fields u and v with values in $\mathbb{R}^3$ we write $u \otimes v$ for the matrix field with the entries $(u \otimes v)_{jm} = u_j v_m$, so that $(u \cdot \nabla)v = \mathrm{div}(v \otimes u)$ if u is divergence-free. With these conventions,

$$\begin{aligned} &e^{\tau\Delta}\mathbb{P}\,\mathrm{div}\, F = \nabla O_\tau * F && (F \in W^{1,1}(\mathbb{R}^3; \mathbb{R}^{3\times 3})), \\ &K(\tau)(u,v) = -\tfrac{1}{2}\, \nabla O_\tau * (u \otimes v + v \otimes u) && (u, v \in X). \end{aligned} \tag{3.10}$$

For the first identity, (3.9) gives $\nabla O_\tau \in L^1$, and (2.2) identifies the Fourier transforms. For the second, first take $u, v \in C^\infty_{c,\mathrm{div}}$ and $F = u \otimes v + v \otimes u$. Pass to X by density: (3.7) and (3.1) control the left side in L^6, while Young's inequality uses $\nabla O_\tau \in L^{6/5}$ and $F \in L^3$ on the right.

3.3. Compactness. At a fixed lag $\tau > 0$, one factor in $K(\tau)$ can be approximated by a compactly supported field. The remaining factor is then needed only on a bounded ball, where Rellich compactness applies. This makes the monodromy operator of Subsection 4.3 a compact perturbation of a contraction.

Lemma 3.2. *For every $u \in X$ and every $\tau > 0$ the operator*

$$v \mapsto K(\tau)(u,v)$$

on X is compact.

Proof. Fix $\tau > 0$. By (3.7), approximating u from $C^\infty_{c,\mathrm{div}}$ approximates $K(\tau)(u,\cdot)$ in operator norm. Compact operators are closed in that norm. It is thus enough to prove $\|K(\tau)(\varphi, v_k)\|_{\dot{H}^1} \to 0$ for $\varphi \in C^\infty_{c,\mathrm{div}}$ and $v_k \rightharpoonup 0$ in X.

Let B be a ball that contains the support of φ. For $v \in X$ the matrix field $F_v = \varphi \otimes v + v \otimes \varphi$ vanishes outside B, and $\mathrm{div}\, F_v = (v \cdot \nabla)\varphi + (\varphi \cdot \nabla)v$, because v and φ are divergence-free; thus

$K(\tau)(\varphi, v) = -\frac{1}{2}\, e^{\tau\Delta}\mathbb{P}\,\mathrm{div}\, F_v$. Since $|\widehat{\mathrm{div}\, F_v}(\xi)| \le 2\pi|\xi|\,|\hat{F}_v(\xi)|$ and $16\pi^4|\xi|^4 e^{-8\pi^2\tau|\xi|^2} \le \tau^{-2}$, the definition (2.2), the last identity of (3.2) and Plancherel's identity give

$$\|K(\tau)(\varphi, v)\|_{\dot{H}^1}^2 = \frac{1}{4}\int 4\pi^2|\xi|^2 e^{-8\pi^2\tau|\xi|^2}\,|\mathbb{P}_\xi \widehat{\mathrm{div}\, F_v}(\xi)|^2\, d\xi \le \frac{\|F_v\|_{L^2}^2}{4\,\tau^2} \le \frac{\|\varphi\|_{L^\infty}^2 \|v\|_{L^2(B)}^2}{\tau^2}.$$

Only the values on B enter the last bound. By (3.1) and $\|v\|_{L^2(B)} \le |B|^{1/3}\|v\|_6$, restriction to B is continuous from X to $H^1(B)$, so $v_k \rightharpoonup 0$ there. Rellich's theorem on the bounded ball gives compactness in $L^2(B)$ [14, Theorem 9.16, p. 285]. Every subsequence thus has a further subsequence converging strongly to its weak limit, zero. The last bound now gives convergence of $K(\tau)(\varphi, v_k)$ in X. □

3.4. An abstract setting. These lemmas and Lemmas 4.7–4.8 use a real Banach space $\mathcal{X}$, linear operators $e^{t\Delta}$ for $t \ge 0$, and symmetric bilinear maps $K(\tau)$ for $\tau > 0$. For every $\chi, u, v \in \mathcal{X}$, assume that $t \mapsto e^{t\Delta}\chi$ and $\tau \mapsto K(\tau)(u, v)$ are continuous on $[0, \infty)$ and $(0, \infty)$, respectively, and that a constant C_K gives

$$e^{0\Delta} = I, \qquad e^{(t+t')\Delta} = e^{t\Delta}e^{t'\Delta}, \qquad \|e^{t\Delta}\chi\|_{\mathcal{X}} \le \|\chi\|_{\mathcal{X}} \qquad (t, t' \ge 0,\ \chi \in \mathcal{X}), \tag{3.11}$$

$$\begin{aligned} e^{t\Delta}K(\tau)(u, v) &= K(t+\tau)(u, v) && (t \ge 0,\ \tau > 0,\ u, v \in \mathcal{X}),\\ \|K(\tau)(u, v)\|_{\mathcal{X}} &\le C_K\,\tau^{-3/4}\,\|u\|_{\mathcal{X}}\,\|v\|_{\mathcal{X}} && (\tau > 0,\ u, v \in \mathcal{X}). \end{aligned} \tag{3.12}$$

On X these properties hold by Subsection 3.1, Lemma 3.1 and (3.8), and on the spaces $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$, $L > 0$, of Subsection 3.6 by Lemma 3.7. Mild solutions in $\mathcal{X}$ are defined by (2.4), with these operators, and the integrals of maps with values in $\mathcal{X}$ are Bochner integrals.

Lemma 3.3. *Let $a \le b$, and let $u, v : [a, b] \to \mathcal{X}$ be continuous. Then for every $t \in [a, b]$ the map $t' \mapsto K(t - t')(u(t'), v(t'))$ is integrable on $[a, t]$, and the map*

$$t \mapsto \int_a^t K(t - t')(u(t'), v(t'))\, dt'$$

is continuous on $[a, b]$.

The next lemma solves the equation linearized about a continuous map f, with additive term g, on an interval of any length; in Section 4 the map f is u^*.

Lemma 3.4. *Let $a \le b$, and let $f : [a, b] \to \mathcal{X}$ be continuous. For every continuous $g : [a, b] \to \mathcal{X}$ there is exactly one continuous $h : [a, b] \to \mathcal{X}$ with*

$$h(t) = 2\int_a^t K(t - t')(f(t'), h(t'))\, dt' + g(t) \qquad (a \le t \le b).$$

The map $g \mapsto h$ is linear, and it is bounded for the supremum norm.

The semigroup property allows the integral equation to be restarted at time m. Applied to a difference of two solutions, the same short-time estimate yields uniqueness.

Lemma 3.5. *Let $a < b$, and let $f, h, u : [a, b] \to \mathcal{X}$ be continuous with $u(t) = e^{(t-a)\Delta}u(a) + \int_a^t K(t - t')(f(t'), h(t'))\, dt'$ for $a \le t \le b$. Then*

$$u(t) = e^{(t-m)\Delta}u(m) + \int_m^t K(t - t')(f(t'), h(t'))\, dt' \qquad (a \le m \le t \le b). \tag{3.13}$$

Moreover, two solutions on $[a, b]$ with the same value at a are equal.

By (3.13) with $f = h = u$, a solution on $[a,b]$ is a solution on $[m,b]$ for $a \le m < b$. Appendix A.1 uses $\int_0^h \tau^{-3/4}\,d\tau = 4h^{1/4}$ for continuity and the weight $e^{-\beta(t-a)}$ for contraction. Splitting the integral at m gives (3.13). For uniqueness apply Lemma 3.4 to $u - v$ with $f = (u+v)/2$ and $g = 0$.

3.5. Solutions in the sense of distributions. Let E_2 be the closure of Schwartz fields for the norm $\sup_x \|f\|_{L^2(x+B_1)}$, where B_1 is the unit ball. For fields in $L^2_t E_2$, Furioli, Lemarié-Rieusset and Terraneo proved the distributional formulation with a pressure equivalent to the integral formulation [55, Definition 2 and Proposition 1, p. 607]; Dubois proved that both are equivalent to formulations without a pressure, tested against divergence-free fields [46, Theorem 1, pp. 29–30]. Testing against solenoidal fields removes ∇p by integration by parts. Dubois's result then recovers the projected integral equation for the velocity in the stated class. Here (3.1), Hölder and density in L^6 give $u \in C([a,b]; E_2)$. Time-dependent testing of (3.14) gives Dubois's weak formulation with datum $u(a)$; Lemma 3.3 extends the integral identity to b (Appendix A.1).

Lemma 3.6. *Let $a < b$, and let $u : [a,b] \to X$ be continuous. Suppose that for every $\psi \in C^\infty_{c,\mathrm{div}}$ the function $t \mapsto \int_{\mathbb{R}^3} u(t)\cdot\psi\,dx$ is continuously differentiable on $[a,b]$, with*

$$\frac{d}{dt}\int_{\mathbb{R}^3} u(t)\cdot\psi\,dx = \int_{\mathbb{R}^3} \big[u(t)\cdot\Delta\psi + (u(t)\otimes u(t)) : \nabla\psi\big]\,dx \qquad (a \le t \le b), \tag{3.14}$$

where $(f\otimes g) : \nabla h = \sum_{i,j} f_i g_j\,\partial_i h_j$ for fields f, g and h on $\mathbb{R}^3$. Then u is a mild solution on $[a,b]$.

3.6. The heat flow and the bilinear term on the torus. Section 6 constructs the solution first at large spatial period and then dilates it to $\mathbb{T}^3$. For $L > 0$, put $\mathbb{T}^3_L = \mathbb{R}^3/L\mathbb{Z}^3$ and $Q_L = [-\frac{L}{2}, \frac{L}{2})^3$; thus $\mathbb{T}^3 = \mathbb{T}^3_1$. Functions on $\mathbb{T}^3_L$ are identified with functions on $\mathbb{R}^3$ having period L in each coordinate. Their L^p norm is taken over Q_L, and their Fourier coefficients are $\hat f(k) = L^{-3}\int_{Q_L} e^{-2\pi i k\cdot x} f(x)\,dx$, $k \in L^{-1}\mathbb{Z}^3$. The space $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$ consists of these periodic divergence-free fields in H^1_{loc} with $\int_{Q_L} u\,dx = 0$; it is a real Hilbert space with norm

$$\|u\|^2_{\dot H^1(\mathbb{T}^3_L)} = \int_{Q_L} |\nabla u|^2\,dx = L^3 \sum_{k\in L^{-1}\mathbb{Z}^3} 4\pi^2|k|^2|\hat u(k)|^2. \tag{3.15}$$

At $L = 1$, mean zero removes $k = 0$, and $|k| \ge 1$ bounds the remaining L^2 coefficients by the gradient coefficients. This gives the equivalent $H^1(\mathbb{T}^3)$ norm used in Theorem 1.1 and the local theory of Tao [84, Section 5; Theorem 5.1, manuscript pp. 28–29]:

$$\|u\|_{\dot H^1(\mathbb{T}^3)} \le \|u\|_{H^1(\mathbb{T}^3)} \lesssim \|u\|_{\dot H^1(\mathbb{T}^3)} \qquad (u \in \dot H^1_{\mathrm{div}}(\mathbb{T}^3)). \tag{3.16}$$

On $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$ the operator $e^{t\Delta}$, $t \ge 0$, multiplies $\hat u(k)$ by $e^{-4\pi^2 t|k|^2}$. For $\tau > 0$ and $f \in L^1(\mathbb{T}^3_L)$, $e^{\tau\Delta}\mathbb{P}f$ is the L-periodic field with the Fourier coefficients $e^{-4\pi^2\tau|k|^2}\,\mathbb{P}_k\hat f(k)$, and for $u, v \in \dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$

$$K_{\mathbb{T}}(\tau)(u,v) = -\tfrac12\, e^{\tau\Delta}\mathbb{P}\big[(u\cdot\nabla)v + (v\cdot\nabla)u\big]. \tag{3.17}$$

Mild solutions in $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$ on $[a,b)$ and on $[a,b]$ are defined by (2.4), with this $e^{t\Delta}$ and with $K_{\mathbb{T}}$ in place of K. We do not show the period in the notation $e^{t\Delta}$, $\mathbb{P}$ and $K_{\mathbb{T}}$.

The heat flow on $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$ has the properties (3.11), and $t \mapsto e^{t\Delta}u$ is continuous on $[0,\infty)$. As on $\mathbb{R}^3$, the bound of the bilinear term rests on the Sobolev inequality

$$\|u\|_{L^6(Q_L)} \lesssim \|u\|_{\dot H^1(\mathbb{T}^3_L)} \qquad (L > 0,\ u \in \dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)), \tag{3.18}$$

with a constant that does not depend on L, and on the smoothing of $e^{\tau\Delta}\mathbb{P}$ from $L^{3/2}$ to $\dot{H}^1$. For $L = 1$, use the mean-zero Sobolev inequality [11, Proposition 1.1, manuscript p. 3], with $s = 1$, $p = 2$, $q = 6$. For $v(z) = u(Lz)$, both norms in (3.18) acquire the factor $L^{1/2}$ relative to v, so the constant is independent of L. For the next lemma set $z = e^{\tau\Delta}\mathbb{P}f$ and $\Psi = -\Delta e^{\tau\Delta}z$. Parseval gives $\int_{Q_L} f \cdot \Psi \, dx = \|z\|^2_{\dot{H}^1}$ (Appendix A.1). Heat smoothing and (3.18) give $\|\Psi\|_2 \lesssim \tau^{-1/2}\|z\|_{\dot{H}^1}$ and $\|\Psi\|_6 \lesssim \tau^{-1}\|z\|_{\dot{H}^1}$. Their geometric mean bounds $\|\Psi\|_3$, so Hölder gives (3.19).

Lemma 3.7. *There exists a constant C such that for all $L > 0$ and $\tau > 0$ and every $f \in L^{3/2}(\mathbb{T}^3_L)$ with $\int_{Q_L} f \, dx = 0$ the field $e^{\tau\Delta}\mathbb{P}f$ belongs to $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ and*

$$\|e^{\tau\Delta}\mathbb{P}f\|_{\dot{H}^1(\mathbb{T}^3_L)} \leq C\,\tau^{-3/4}\,\|f\|_{L^{3/2}(Q_L)}. \tag{3.19}$$

Moreover $e^{t\Delta}e^{\tau\Delta}\mathbb{P}f = e^{(t+\tau)\Delta}\mathbb{P}f$ for $t \geq 0$, and $\tau \mapsto e^{\tau\Delta}\mathbb{P}f$ is continuous from $(0,\infty)$ to $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$. In particular, there exists a constant $C_{\mathbb{T}}$ such that for all $L > 0$, $\tau > 0$ and $u, v \in \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ the field $(u \cdot \nabla)v + (v \cdot \nabla)u$ belongs to $L^{3/2}(\mathbb{T}^3_L)$ and has mean zero, the field $K_{\mathbb{T}}(\tau)(u, v)$ belongs to $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$, and

$$\|K_{\mathbb{T}}(\tau)(u,v)\|_{\dot{H}^1(\mathbb{T}^3_L)} \leq C_{\mathbb{T}}\,\tau^{-3/4}\,\|u\|_{\dot{H}^1(\mathbb{T}^3_L)}\,\|v\|_{\dot{H}^1(\mathbb{T}^3_L)}. \tag{3.20}$$

Thus on each of the spaces $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ the operators $e^{t\Delta}$ and the maps $K_{\mathbb{T}}(\tau)$ have the properties required in Subsection 3.4, with the constant $C_{\mathbb{T}}$, which does not depend on L, in place of C_K.

The dilations carry these objects from one period to another. Let $L, \mu > 0$. The operator D_μ of (3.4) maps $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ onto $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_{L/\sqrt{\mu}})$, and for $u, v \in \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$, $t \geq 0$ and $\tau > 0$

$$\begin{gathered}\|D_\mu u\|_{\dot{H}^1(\mathbb{T}^3_{L/\sqrt{\mu}})} = \mu^{1/4}\,\|u\|_{\dot{H}^1(\mathbb{T}^3_L)}, \qquad D_\mu e^{\mu t\Delta}u = e^{t\Delta}D_\mu u, \\ \mu\, D_\mu K_{\mathbb{T}}(\mu\tau)(u,v) = K_{\mathbb{T}}(\tau)(D_\mu u, D_\mu v).\end{gathered} \tag{3.21}$$

These identities are proved as (3.5) and the second identity of (3.8), with Fourier series in place of the Fourier transform.

4. The equation linearized about u^*

We choose a rate $\kappa \in (0, 1/4)$ by placing the circle $|z| = e^{-S\kappa}$ in the resolvent set of the period map (Lemma 4.5). The associated exterior projection singles out the finitely many directions responsible for growth or slower decay. Its rank and the norm of the inverse in Proposition 4.3 can depend on the circle. Lemma 4.8 combines the resulting right inverse with a bilinear estimate to solve the nonlinear equation by contraction.

Throughout this section (W, q) is a profile with period $S > 0$ and isometry R, C_W is the constant of (1.3), $\lambda = e^{-S} \in (0, 1)$, and $u^* = u^*_1$ is (1.4) with $T = 1$. Unless specified otherwise, the constants in $\lesssim$ depend only on this profile, independently of t and of the period index n.

4.1. The field u^*. In the notation (3.4) the field (1.4) of the profile is $u^*_T(t) = D_{1/(T-t)}W(s, \cdot)$ with $s = \log(T/(T-t))$. The field u^* is a smooth function of $(t, x) \in (-\infty, 1) \times \mathbb{R}^3$, divergence-free in x, and by (1.3)

$$|u^*(t,x)| \leq \frac{C_W}{\sqrt{1-t}+|x|} \leq \frac{C_W}{|x|}, \qquad |\nabla u^*(t,x)| \leq \frac{C_W}{\left(\sqrt{1-t}+|x|\right)^2} \leq \frac{C_W}{|x|^2}. \tag{4.1}$$

At fixed $t < 1$ both bounds are finite at the origin. At infinity their sixth and second powers, respectively, are integrable, so $u^*(t) \in L^6$ and $\nabla u^*(t) \in L^2$. In particular $u^*(t) \in X$.

Proposition 4.1. *Let (W, q) be a profile with period S and isometry R, and let $\lambda = e^{-S}$. Then the field $u^* = u_1^*$ of (1.4), restricted to $[0,1)$, is a continuous map from $[0,1)$ to X, not identically zero, which is a mild solution on $[0,1)$ and satisfies*

$$u^*(1-\lambda(1-t)) = \rho_R\, D_{1/\lambda}\, u^*(t) \qquad (0 \le t < 1). \tag{4.2}$$

Proof. Set $u = u_1^*$ and $p(t,x) = (1-t)^{-1} q(s,y)$, with $s = -\log(1-t)$ and $y = x/\sqrt{1-t}$. The change of variables transforms the first two equations of (1.2) into (1.1), with u, p smooth on $(-\infty, 1) \times \mathbb{R}^3$. Indeed, $\partial_t u = (1-t)^{-3/2}[\partial_s W + \frac{1}{2} W + \frac{1}{2}(y\cdot\nabla)W](s,y)$; the diffusion, advection and pressure-gradient terms have the same factor. Fix $0 < b < 1$. Smoothness gives pointwise continuity of ∇u in time, while (4.1) bounds it by $C_W(\sqrt{1-b}+|x|)^{-2} \in L^2$, uniformly on $[0,b]$. Dominated convergence therefore gives continuity into X. Testing (1.1) against $\psi \in C^\infty_{c,\mathrm{div}}$ gives (3.14): compact support removes boundary terms, and $\operatorname{div}\psi = 0$ removes the pressure. Lemma 3.6 yields mildness on $[0,b]$, hence on $[0,1)$ as $b < 1$ is arbitrary. Relation (1.5), with $T = 1$, is (4.2). If u^* vanished on $[0,1)$, then W would vanish for $s \ge 0$; the recurrence in (1.2) extends this to every $s \in \mathbb{R}$, contradicting the nonvanishing of a profile. □

In (4.2) the time $1-\lambda(1-t)$ is the time t advanced by the period S in the self-similar time $s = -\log(1-t)$. We write $t_n = 1-\lambda^n$ for $n \ge 0$, so that $[t_n, t_{n+1}]$ is the physical interval corresponding to $nS \le s \le (n+1)S$. Its length is $\lambda^n(1-\lambda)$, and the background returns to its earlier form after undoing the dilation and isometry in (4.2). The next lemma expresses u^* on each period through its restriction to the first, and it gives the rate of u^*.

Lemma 4.2. *For $n \ge 0$ and $0 \le \sigma < 1$,*

$$u^*(t_n + \lambda^n \sigma) = \rho_R^n\, D_{\lambda^{-n}}\, u^*(\sigma). \tag{4.3}$$

There exist constants $0 < c_* \le C$ such that

$$c_*\,(1-t)^{-1/4} \le \|u^*(t)\|_{\dot H^1} \le C\,(1-t)^{-1/4} \qquad (0 \le t < 1). \tag{4.4}$$

Proof. For $t = t_n + \lambda^n\sigma$ we have $1-\lambda(1-t) = t_{n+1} + \lambda^{n+1}\sigma$, so that (4.2) reads $u^*(t_{n+1} + \lambda^{n+1}\sigma) = \rho_R\, D_{1/\lambda}\, u^*(t_n + \lambda^n\sigma)$. Since ρ_R commutes with the dilations, (4.3) follows by induction on n.

Every $t \in [0,1)$ is $t_n + \lambda^n\sigma$ for some $n \ge 0$ and $\sigma \in [0, t_1)$. Then $\lambda^{n+1} < 1 - t \le \lambda^n$, and $\|u^*(t)\|_{\dot H^1} = \lambda^{-n/4}\,\|u^*(\sigma)\|_{\dot H^1}$ by (4.3) and (3.5). Hence (4.4) holds with

$$C = \max_{[0,t_1]} \|u^*\|_{\dot H^1}, \qquad c_* = \lambda^{1/4} \min_{[0,t_1]} \|u^*\|_{\dot H^1},$$

provided u^* vanishes at no time. If $u^*(a) = 0$, uniqueness in Lemma 3.5 gives $u^* = 0$ on every $[a,b]$, $a < b < 1$. Every earlier time advances into $[a,1)$ under (4.3); since its dilation and isometry are invertible, u^* would vanish everywhere on $[0,1)$, contrary to Proposition 4.1. Continuity on $[0,t_1]$ then makes the displayed minimum positive. □

4.2. The linearized equation. Let $0 \le \kappa < \frac{1}{4}$. For a Banach space $\mathcal{X}$, $C_\kappa([0,1);\mathcal{X})$ is the space of continuous maps $v : [0,1) \to \mathcal{X}$ with

$$\|v\|_\kappa = \sup_{0\le t<1} (1-t)^{\frac{1}{4}-\kappa}\,\|v(t)\|_{\mathcal{X}} < \infty. \tag{4.5}$$

Multiplication by $(1-t)^{1/4-\kappa}$ identifies this space isometrically with the bounded continuous $\mathcal{X}$-valued maps on $[0,1)$, so it is complete. Since $1-t \le 1$, we have $\|v\|_0 \le \|v\|_\kappa \le \sup_{0\le t<1}\|v(t)\|_{\mathcal{X}}$. For continuous $f, g : [0,1) \to X$ the map $\mathcal{B}(f,g)$ from $[0,1)$ to X is

$$\mathcal{B}(f,g)(t) = \int_0^t K(t-t')(f(t'), g(t'))\,dt'. \tag{4.6}$$

For $\chi \in X$ we write $e^{t\Delta}\chi$ also for the map $t \mapsto e^{t\Delta}\chi$, in the norms $\|\cdot\|_\kappa$ and in identities between maps. In the setting of Subsection 3.4 the same notation is used with $\mathcal{X}$ in place of X.

In the norm (4.5) the factor $(1-t)^{1/4}$ is the inverse of the rate (4.4) of u^*, and the other factor is $e^{\kappa s}$ in the self-similar time $s = -\log(1-t)$: up to a constant, a map in $C_\kappa([0,1);X)$ is smaller than u^* by the factor $e^{-\kappa s}$. For a continuous $u : [0,1) \to X$ the equation (2.4) on $[0,1)$ reads $u = e^{t\Delta}u(0) + \mathcal{B}(u,u)$. Since u^* satisfies it and $\mathcal{B}$ is symmetric, $u^* + v$ satisfies it, for a continuous $v : [0,1) \to X$, if and only if

$$v = e^{t\Delta}v(0) + 2\,\mathcal{B}(u^*, v) + \mathcal{B}(v,v). \tag{4.7}$$

Let g denote the additive term in the mild equation (4.8). The endpoint values propagated from one period to the next are those of $v - g$. Returning them to the scale and orientation of the first period gives a sequence that must decay like Θ^n, where $\Theta = \lambda^\kappa \in (\lambda^{1/4}, 1)$. On the spectral subspace inside $|z| = \Theta$, forward iteration gives this decay. On the finite-dimensional exterior subspace, the required decay prescribes the initial projection in terms of all future inhomogeneous terms. This prescription also accounts for modes with $\Theta < |z| < 1$.

Proposition 4.3. *There exist $\kappa \in (0, \frac{1}{4})$, a finite-dimensional subspace E of $C^\infty_{c,\mathrm{div}}$, a constant $C_\mathcal{R}$ and a linear map $\mathcal{R} : g \mapsto (\gamma, v)$ from $C_\kappa([0,1);X)$ to $E \times C_\kappa([0,1);X)$ with*

$$\begin{gathered} v(t) - 2\,\mathcal{B}(u^*, v)(t) - e^{t\Delta}\gamma = g(t) \qquad (0 \le t < 1), \\ \max\{\|\gamma\|_{\dot H^1}, \|v\|_\kappa\} \le C_\mathcal{R}\,\|g\|_\kappa. \end{gathered} \tag{4.8}$$

4.3. The monodromy operator. For continuous maps on $[0,t_1]$ we keep the notation $\mathcal{B}(f,g)$ of (4.6). By Lemma 3.4 with $f = u^*$, for every continuous $g : [0,t_1] \to X$ there is exactly one continuous $h : [0,t_1] \to X$ with

$$h = 2\,\mathcal{B}(u^*, h) + g \qquad \text{on } [0,t_1];$$

we write $h = \mathcal{U}g$. The operator $\mathcal{U}$ is linear and bounded for the supremum norm on $[0,t_1]$, and we denote its norm by $\|\mathcal{U}\|$. For $\chi \in X$ let $h_\chi = \mathcal{U}(e^{t\Delta}\chi)$, the solution of the equation linearized about u^* with the datum χ:

$$h_\chi(t) = e^{t\Delta}\chi + 2\int_0^t K(t-t')(u^*(t'), h_\chi(t'))\,dt' \qquad (0 \le t \le t_1). \tag{4.9}$$

At the end of the first physical interval, (4.3) has changed the scale and orientation of the background. The *monodromy operator* evolves χ to t_1 and undoes these changes,

$$\mathcal{M}\chi = \rho_R^{-1}\,D_\lambda\,h_\chi(t_1) \qquad (\chi \in X). \tag{4.10}$$

Thus in similarity variables $\mathcal{M}$ is one period of linearized evolution followed by ρ_R^{-1}. The isometry preserves the X norm and D_λ multiplies it by $\lambda^{1/4}$. Since heat flow is contractive, $\|\mathcal{M}\| \le \lambda^{1/4}\|\mathcal{U}\|$ by (3.3)–(3.5).

Lemma 4.4. *The operator $\mathcal{M} - \rho_R^{-1}D_\lambda e^{t_1\Delta}$ on X is compact, and*

$$\|\rho_R^{-1}D_\lambda e^{t_1\Delta}\| \le \lambda^{1/4} = e^{-S/4}.$$

Proof. The bound holds because ρ_R^{-1} is an isometry, $\|D_\lambda\chi\|_{\dot H^1} = \lambda^{1/4}\|\chi\|_{\dot H^1}$ and $\|e^{t_1\Delta}\chi\|_{\dot H^1} \le \|\chi\|_{\dot H^1}$, by (3.5) and (3.3). By (4.9) at time t_1 and by (4.10),

$$\mathcal{M}\chi - \rho_R^{-1}D_\lambda e^{t_1\Delta}\chi = 2\,\rho_R^{-1}D_\lambda\int_0^{t_1} K(t_1-t')(u^*(t'), h_\chi(t'))\,dt' \qquad (\chi \in X).$$

By the Hilbert-space compactness criterion used in Lemma 3.2, it suffices that the integral tend to 0 in X for every weakly null sequence χ_k. Such a sequence is bounded. For each $t' \in [0,t_1]$ the map $\chi \mapsto h_\chi(t')$ is linear and bounded, with norm at most $\|\mathcal{U}\|$, so that $h_{\chi_k}(t') \to 0$ weakly. For $t' < t_1$ the operator $K(t_1-t')(u^*(t'),\cdot)$ is compact (Lemma 3.2), so that the integrand tends to 0 in X at every $t' < t_1$. By (3.7) its norm is at most $C_K\,(t_1 - t')^{-3/4}\,\max_{[0,t_1]}\|u^*\|_{\dot H^1}\,\|\mathcal{U}\|\,\sup_k\|\chi_k\|_{\dot H^1}$, which is integrable on $[0,t_1]$, and the integral tends to 0 by dominated convergence. □

The compact difference in Lemma 4.4 leaves only finitely many modes outside a suitable spectral circle. A gap on that circle gives summable power bounds on both spectral subspaces.

Lemma 4.5. *There exist a number $\Theta \in (e^{-S/4},1)$ and a bounded projection P_{u} of finite rank on X that commutes with $\mathcal{M}$, such that the restriction $\mathcal{M}_{\mathrm{u}}$ of $\mathcal{M}$ to $\operatorname{ran} P_{\mathrm{u}}$ is invertible and*

$$\sum_{n\ge 0}\Theta^{-n}\,\|\mathcal{M}^n(I-P_{\mathrm{u}})\| + \sum_{n\ge 0}\Theta^{n}\,\|\mathcal{M}_{\mathrm{u}}^{-n}P_{\mathrm{u}}\| < \infty, \tag{4.11}$$

where the norms are operator norms on X.

Proof. We complexify X with $\|x+iy\|^2 = \|x\|^2+\|y\|^2$. Extending each real operator componentwise preserves its norm, and preserves compactness by applying it to the real and imaginary parts. Thus $\mathcal{M}$ and $\rho_R^{-1}D_\lambda e^{t_1\Delta}$ retain their norms and compact difference. Lemma 4.4 compares one period of the linearized flow with the free heat flow. Buckmaster, Coiculescu and Li use a compact difference of evolution operators over one period in their forward problem [17, §19, proof of Proposition 19.1]. Here the norm bound for the free operator is $e^{-S/4}$. Compact perturbations preserve the essential spectral radius, so outside this disk the spectrum of $\mathcal{M}$ consists of isolated poles of finite algebraic multiplicity [51, Chapter IV, Paragraph 1.20, pp. 248–249, and (1.16), p. 249]. For each $r > e^{-S/4}$, compactness of the spectrum allows only finitely many such points with $|z| \ge r$. We can therefore choose $\Theta \in (e^{-S/4},1)$ so that no spectral point lies on $|z| = \Theta$.

Let P be the Riesz projection for the spectrum of $\mathcal{M}$ in $\{|z| > \Theta\}$ [51, Chapter IV, (1.6)–(1.8), p. 244]. It commutes with $\mathcal{M}$ and has finite rank, the sum of the finite algebraic multiplicities [51, Chapter IV, Paragraph 1.17, pp. 246–247]. Complex conjugation commutes with $\mathcal{M}$ and preserves this spectral set; conjugating the resolvent integral therefore leaves P unchanged. Hence P preserves the real space X; denote its restriction by P_{u}. The subscript labels every mode exterior to $|z| = \Theta$, including the slowly decaying ones. The spectra of the restrictions of $\mathcal{M}$ to $\operatorname{ran}(I-P)$ and to $\operatorname{ran} P$ are the parts of the spectrum of $\mathcal{M}$ in $\{|z| < \Theta\}$ and in $\{|z| > \Theta\}$. Hence the first restriction has spectral radius less than Θ, and the second is invertible, with an inverse of spectral radius less than Θ^{-1}; in particular $\mathcal{M}_{\mathrm{u}}$ is invertible. The spectral radius formula [51, Chapter III, Exercise 1.17(2), p. 167] gives geometric bounds with ratios $r_{\mathrm{s}} < \Theta$ for forward powers and $r_{\mathrm{b}} < \Theta^{-1}$ for inverse powers. After the weights in (4.11), their ratios are $r_{\mathrm{s}}/\Theta < 1$ and $\Theta r_{\mathrm{b}} < 1$, so both series converge. □

4.4. Proof of Proposition 4.3. In this subsection Θ, P_{u} and $\mathcal{M}_{\mathrm{u}}$ are those of Lemma 4.5, and κ is defined by $\lambda^\kappa = \Theta$; then $0 < \kappa < \frac14$, because $\lambda^{1/4} < \Theta < 1$. The next lemma restricts the inhomogeneous mild equation to the intervals $[t_n, t_{n+1}]$ and rescales each of them to $[0,t_1]$. The additive term g is a trajectory in the integral equation. Subtracting g before restarting gives the heat evolution of $(v-g)(t_n)$ on the next interval. The same background u^* then appears on every rescaled interval by (4.3).

Lemma 4.6. *Let $g \in C_\kappa([0,1);X)$ and $\chi \in X$, and let $v : [0,1) \to X$ be the continuous map with*

$$v(t) = e^{t\Delta}\chi + 2\,\mathcal{B}(u^*, v)(t) + g(t) \qquad (0 \le t < 1). \tag{4.12}$$

For $n \ge 0$ and $0 \le \sigma \le t_1$ put

$$\begin{gathered} v_n(\sigma) = \rho_R^{-n}\, D_{\lambda^n}\, v(t_n + \lambda^n\sigma), \qquad g_n(\sigma) = \rho_R^{-n}\, D_{\lambda^n}\, g(t_n + \lambda^n\sigma), \\ \chi_n = v_n(0) - g_n(0), \qquad f_n = \rho_R^{-1} D_\lambda\, (\mathcal{U} g_n - g_n)(t_1). \end{gathered} \tag{4.13}$$

Then $\chi_0 = \chi$, and for every $n \ge 0$

$$v_n = h_{\chi_n} + \mathcal{U} g_n, \qquad \chi_{n+1} = \mathcal{M}\chi_n + f_n, \tag{4.14}$$

$$\max_{[0,t_1]} \|g_n\|_{\dot H^1} + \|f_n\|_{\dot H^1} \le C\,\Theta^n\,\|g\|_\kappa, \tag{4.15}$$

with a constant C that does not depend on n, g and χ. The sequence (f_n) depends linearly on g, and it does not depend on χ.

Proof. Lemma 3.4, with $f = u^*$, gives a continuous solution of (4.12) on each $[0,b]$, $b < 1$. Uniqueness makes these solutions agree on overlaps, giving the unique solution on $[0,1)$. The map $v - g$ has the value χ at time 0, so that $\chi_0 = \chi$. By (4.12) and by (3.13), applied on the same intervals with $u = v - g$, $f = 2u^*$ and $h = v$,

$$(v-g)(t) = e^{(t-t_n)\Delta}(v-g)(t_n) + 2\int_{t_n}^{t} K(t-t')(u^*(t'), v(t'))\,dt' \qquad (t_n \le t < 1).$$

We apply $\rho_R^{-n} D_{\lambda^n}$ to this identity at time $t = t_n + \lambda^n\sigma$. On the left we obtain $v_n(\sigma) - g_n(\sigma)$. In the first term on the right, $\rho_R^{-n} D_{\lambda^n} e^{\lambda^n\sigma\Delta} = e^{\sigma\Delta}\rho_R^{-n} D_{\lambda^n}$, by (3.5) and since ρ_R commutes with the heat flow and with the dilations; thus this term becomes $e^{\sigma\Delta}\chi_n$. In the integral we substitute $t' = t_n + \lambda^n\sigma'$. By the last two identities of (3.8) and by (4.3), in the form $\rho_R^{-n} D_{\lambda^n} u^*(t_n + \lambda^n\sigma') = u^*(\sigma')$,

$$\lambda^n \rho_R^{-n} D_{\lambda^n} K(\lambda^n(\sigma - \sigma'))(u^*(t'), v(t')) = K(\sigma - \sigma')(u^*(\sigma'), v_n(\sigma')) \qquad (0 \le \sigma' < \sigma),$$

so that the integral becomes $\mathcal{B}(u^*, v_n)(\sigma)$. Therefore

$$v_n = e^{t\Delta}\chi_n + 2\mathcal{B}(u^*, v_n) + g_n \quad \text{on } [0,t_1], \qquad \text{that is,} \qquad v_n = \mathcal{U}(e^{t\Delta}\chi_n + g_n) = h_{\chi_n} + \mathcal{U} g_n.$$

Since $t_n + \lambda^n t_1 = t_{n+1}$, we have $v_{n+1}(0) = \rho_R^{-1} D_\lambda v_n(t_1)$ and $g_{n+1}(0) = \rho_R^{-1} D_\lambda g_n(t_1)$, so that

$$\chi_{n+1} = \rho_R^{-1} D_\lambda (h_{\chi_n} + \mathcal{U} g_n - g_n)(t_1) = \mathcal{M}\chi_n + f_n,$$

by (4.10) and the definition (4.13) of f_n.

For $t = t_n + \lambda^n\sigma$, $0 \le \sigma \le t_1$, we have $1 - t \ge \lambda^{n+1}$. Since $\kappa - \frac{1}{4} < 0$, (4.5) gives

$$\|g_n(\sigma)\|_{\dot H^1} = \lambda^{n/4}\|g(t)\|_{\dot H^1} \le \lambda^{n/4}\lambda^{(n+1)(\kappa-\frac14)}\|g\|_\kappa = \lambda^{\kappa-\frac14}\Theta^n\|g\|_\kappa.$$

This bounds the first term of (4.15). By the definition of f_n the second term is at most $\lambda^{1/4}(1 + \|\mathcal{U}\|)$ times the first, because $\rho_R^{-1} D_\lambda$ multiplies the norm of X by $\lambda^{1/4}$. □

Proof of Proposition 4.3. The spectral projection prescribes $P_{\mathrm{u}}\gamma$, whereas Sections 5 and 6 require γ itself to have compact support. We can meet both requirements. Since $C^\infty_{c,\mathrm{div}}$ is dense and P_{u} is continuous, its image on this class is dense in $\operatorname{ran} P_{\mathrm{u}}$. This linear image is closed by finite dimensionality, so every basis vector in the range has a compactly supported smooth solenoidal preimage. Their span E makes $P_{\mathrm{u}}|_E$ an isomorphism. Its bounded inverse gives $\|\gamma\|_{\dot H^1} \lesssim \|P_{\mathrm{u}}\gamma\|_{\dot H^1}$. Constants may depend on the fixed choices of Θ, κ, P_{u} and E.

Fix $g \in C_\kappa([0,1);X)$. Its forcing terms f_m are independent of the datum (Lemma 4.6). In (4.14), $P_{\mathrm{u}} f_m$ first contributes to χ_{m+1}. Pull it back $m+1$ periods to time zero. By (4.15) the resulting term has norm $\lesssim \Theta^{m+1}\|\mathcal{M}_{\mathrm{u}}^{-(m+1)}P_{\mathrm{u}}\|\,\|g\|_\kappa$, so these terms are summable by (4.11). The Lyapunov–Perron choice cancels their sum with $P_{\mathrm{u}}\gamma$. Since $P_{\mathrm{u}}|_E$ is an isomorphism, this determines exactly one $\gamma \in E$ with

$$P_{\mathrm{u}}\gamma = -\sum_{m\ge 0} \mathcal{M}_{\mathrm{u}}^{-(m+1)} P_{\mathrm{u}} f_m, \qquad \text{and} \qquad \|\gamma\|_{\dot H^1} \lesssim \|g\|_\kappa.$$

We take $\chi = \gamma$ in Lemma 4.6. Since P_{u} commutes with $\mathcal{M}$, the recursion gives, by induction on n,

$$P_{\mathrm{u}}\chi_n = -\sum_{m\ge n} \mathcal{M}_{\mathrm{u}}^{-(m-n+1)} P_{\mathrm{u}} f_m, \qquad (I-P_{\mathrm{u}})\chi_n = \mathcal{M}^n(I-P_{\mathrm{u}})\gamma + \sum_{m<n} \mathcal{M}^{n-1-m}(I-P_{\mathrm{u}})f_m.$$

We bound the terms of the two sums by (4.15), with $m = n-1+j$ in the first sum and $m = n-1-j$ in the second:

$$\|P_{\mathrm{u}}\chi_n\|_{\dot H^1} \lesssim \Theta^{n-1}\|g\|_\kappa \sum_{j\ge 1} \Theta^j \|\mathcal{M}_{\mathrm{u}}^{-j}P_{\mathrm{u}}\|,$$

$$\|(I-P_{\mathrm{u}})\chi_n\|_{\dot H^1} \lesssim \|\mathcal{M}^n(I-P_{\mathrm{u}})\|\,\|\gamma\|_{\dot H^1} + \Theta^{n-1}\|g\|_\kappa \sum_{0\le j<n} \Theta^{-j}\|\mathcal{M}^j(I-P_{\mathrm{u}})\|.$$

By (4.11) the two sums are bounded and $\|\mathcal{M}^n(I-P_{\mathrm{u}})\| \lesssim \Theta^n$. Thus $\|\chi_n\|_{\dot H^1} \lesssim \Theta^n\|g\|_\kappa$ for every $n \ge 0$.

We return to v. By the first identity of (4.14), by (4.15) and by the bound for χ_n,

$$\max_{[0,t_1]} \|v_n\|_{\dot H^1} \le \|\mathcal{U}\| \left(\|\chi_n\|_{\dot H^1} + \max_{[0,t_1]} \|g_n\|_{\dot H^1} \right) \lesssim \Theta^n \|g\|_\kappa \qquad (n \ge 0).$$

For $t = t_n + \lambda^n\sigma$ with $0 \le \sigma \le t_1$ we have $1 - t \le \lambda^n$ and $\lambda^{-n/4}\Theta^n = \lambda^{n(\kappa-\frac14)}$, hence

$$\|v(t)\|_{\dot H^1} = \lambda^{-n/4}\|v_n(\sigma)\|_{\dot H^1} \lesssim \lambda^{n(\kappa-\frac14)}\|g\|_\kappa \le (1-t)^{\kappa-\frac14}\|g\|_\kappa.$$

Thus $v \in C_\kappa([0,1);X)$ with $\|v\|_\kappa \lesssim \|g\|_\kappa$, and (4.12) with $\chi = \gamma$ is the equation of (4.8). The map $\mathcal{R} : g \mapsto (\gamma, v)$ is linear, because γ depends linearly on (f_n) and the solution of (4.12) is unique. □

4.5. Correction of an approximate solution. Section 5 changes the initial velocity on $\mathbb{R}^3$. The small input to the contraction is the heat evolution of the initial mismatch, and Proposition 4.3 supplies the inverse. In Section 6 the small input is the residual of the periodized field, and Subsections 6.3–6.4 construct the inverse by correcting a whole-space approximation with a Neumann series. Both constructions use the contraction below, in the setting of Subsection 3.4. It requires only the weighted bound on the trajectory component of the inverse.

Let $0 < \kappa < \frac14$. For a continuous map $u^{\mathrm{a}} : [0,1) \to \mathcal{X}$, a vector space E and a linear map $J : E \to \mathcal{X}$ we write

$$\begin{aligned} F^{\mathrm{a}}(t) &= u^{\mathrm{a}}(t) - e^{t\Delta}u^{\mathrm{a}}(0) - \mathcal{B}(u^{\mathrm{a}},u^{\mathrm{a}})(t), \\ \mathcal{L}(\gamma,v)(t) &= v(t) - 2\mathcal{B}(u^{\mathrm{a}},v)(t) - e^{t\Delta}J\gamma \end{aligned} \qquad (0 \le t < 1) \tag{4.16}$$

for $\gamma \in E$ and $v \in C_\kappa([0,1);\mathcal{X})$. The map F^{a} is the *residual* of u^{a}: it vanishes if and only if u^{a} is a solution on $[0,1)$. Let $\chi \in \mathcal{X}$, and let $u = u^{\mathrm{a}} + v$. Since $\mathcal{B}$ is bilinear and symmetric, the definitions (4.16) give

$$u - e^{t\Delta}(u^{\mathrm{a}}(0) + \chi + J\gamma) - \mathcal{B}(u,u) = F^{\mathrm{a}} + \mathcal{L}(\gamma,v) - e^{t\Delta}\chi - \mathcal{B}(v,v).$$

Thus u is a solution on $[0,1)$ with the datum $u^{\mathrm{a}}(0)+\chi+J\gamma$ if and only if

$$\mathcal{L}(\gamma,v)=e^{t\Delta}\chi-F^{\mathrm{a}}+\mathcal{B}(v,v). \tag{4.17}$$

Since $F^{\mathrm{a}}(0)=0$, equation (4.17) at $t=0$ gives $v(0)=\chi+J\gamma$. Here χ is prescribed, while γ is found together with v. For $u^{\mathrm{a}}=u^*$ and the inclusion J of the space E of Proposition 4.3 in X, the equation (4.17) is (4.7) with $v(0)=\chi+\gamma$.

We solve (4.17) by a contraction, given a right inverse of $\mathcal{L}$. The next lemma bounds the terms $\mathcal{B}(v,v)$ and $e^{t\Delta}\chi$ of its right-hand side in the norms (4.5). The bounds for $\mathcal{B}$ rest on one estimate: for $0\le t<1$ the substitution $t-t'=(1-t)\theta$ gives

$$\int_0^t (t-t')^{-3/4}(1-t')^{\kappa-\frac12}\,dt' = (1-t)^{\kappa-\frac14}\int_0^{t/(1-t)} \theta^{-3/4}(1+\theta)^{\kappa-\frac12}\,d\theta \le C(1-t)^{\kappa-\frac14}, \tag{4.18}$$

where C is that integral over $(0,\infty)$. Near zero the integrand is bounded by a constant times $\theta^{-3/4}$; at infinity it is bounded by a constant times $\theta^{\kappa-5/4}$. At $\kappa=1/4$ the integrand is asymptotic to θ^{-1} at infinity. The logarithmic divergence explains why this estimate stops short of the endpoint, where the spectral circle $e^{-\kappa S}$ also meets the free bound $e^{-S/4}$.

Lemma 4.7. *Let $0<\kappa<\frac14$. There exists a constant C, which depends only on κ and on the constant C_K of (3.12), such that for all $v,v'\in C_\kappa([0,1);\mathcal{X})$, all $\chi\in\mathcal{X}$, all $g\in C_0([0,1);\mathcal{X})$ and every bounded continuous $f:[0,1)\to\mathcal{X}$ the maps $\mathcal{B}(v,v')$ and $\mathcal{B}(f,g)$ are continuous on $[0,1)$ and*

$$\|\mathcal{B}(v,v')\|_\kappa \le C\,\|v\|_\kappa\,\|v'\|_\kappa, \tag{4.19}$$

$$\|e^{t\Delta}\chi\|_\kappa \le \|\chi\|_{\mathcal{X}}, \tag{4.20}$$

$$\|\mathcal{B}(f,g)\|_\kappa \le C\sup_{0\le t<1}\|f(t)\|_{\mathcal{X}}\,\|g\|_0. \tag{4.21}$$

Proof. The maps $\mathcal{B}(v,v')$ and $\mathcal{B}(f,g)$ are continuous on $[0,1)$ by Lemma 3.3, applied on $[0,b]$ for every $b<1$. Let $0\le t'<t<1$. By (3.12) the norm of $K(t-t')$ is at most $C_K(t-t')^{-3/4}$, and by (4.5)

$$\|v(t')\|_{\mathcal{X}}\,\|v'(t')\|_{\mathcal{X}} \le \|v\|_\kappa\,\|v'\|_\kappa(1-t')^{2\kappa-\frac12}, \qquad \|g(t')\|_{\mathcal{X}} \le \|g\|_0(1-t')^{-\frac14}.$$

Since $0<1-t'\le 1$ and $0<\kappa<\frac14$, both powers of $1-t'$ are at most $(1-t')^{\kappa-\frac12}$. Hence (4.18) gives (4.19) and (4.21), with C_K times the constant of (4.18). Finally (4.20) follows from the last bound of (3.11). □

Lemma 4.8. *Let $0<\kappa<\frac14$. There exists $\varepsilon_1>0$, which depends only on κ and on the constant C_K of (3.12), with the following property. Let $u^{\mathrm{a}}:[0,1)\to\mathcal{X}$ be continuous, let E be a vector space and $J:E\to\mathcal{X}$ a linear map, and let F^{a} and $\mathcal{L}$ be as in (4.16). Suppose that $F^{\mathrm{a}}\in C_\kappa([0,1);\mathcal{X})$ and that there are a constant $C_{\mathcal{R}}$ and a linear map $\mathcal{R}:g\mapsto(\gamma,v)$ from $C_\kappa([0,1);\mathcal{X})$ to $E\times C_\kappa([0,1);\mathcal{X})$ such that*

$$\mathcal{L}(\gamma,v)=g, \qquad \|v\|_\kappa\le C_{\mathcal{R}}\|g\|_\kappa \qquad (g\in C_\kappa([0,1);\mathcal{X})). \tag{4.22}$$

Let $\chi\in\mathcal{X}$ be such that

$$C_{\mathcal{R}}^2\|e^{t\Delta}\chi-F^{\mathrm{a}}\|_\kappa\le\varepsilon_1. \tag{4.23}$$

Then there are $\gamma\in E$ and $v\in C_\kappa([0,1);\mathcal{X})$ which satisfy (4.17) and

$$\|v\|_\kappa\le 2\,C_{\mathcal{R}}\|e^{t\Delta}\chi-F^{\mathrm{a}}\|_\kappa. \tag{4.24}$$

In particular $u^{\mathrm{a}}+v$ is a solution on $[0,1)$ with the value $u^{\mathrm{a}}(0)+\chi+J\gamma$ at time 0.

Proof. Let C be the constant of Lemma 4.7, and let $\varepsilon_1 = (8C)^{-1}$. By (4.20), $e^{t\Delta}\chi \in C_\kappa([0,1);\mathcal{X})$. Let $\delta = 2\,C_{\mathcal{R}}\|e^{t\Delta}\chi - F^{\mathrm{a}}\|_\kappa$, so that $2C\,C_{\mathcal{R}}\delta \le \frac12$ by (4.23). For v' in the closed ball of radius δ of $C_\kappa([0,1);\mathcal{X})$ let $\Phi(v')$ be the second component of

$$\mathcal{R}\big[e^{t\Delta}\chi - F^{\mathrm{a}} + \mathcal{B}(v',v')\big].$$

By (4.22) and (4.19),

$$\|\Phi(v')\|_\kappa \le C_{\mathcal{R}}\|e^{t\Delta}\chi - F^{\mathrm{a}}\|_\kappa + C\,C_{\mathcal{R}}\delta^2 \le \frac{\delta}{2} + \frac{\delta}{4},$$

and since $\mathcal{R}$ is linear and $\mathcal{B}(v',v') - \mathcal{B}(v'',v'') = \mathcal{B}(v'-v'',v') + \mathcal{B}(v'',v'-v'')$, the map Φ is Lipschitz on the ball with constant $2C\,C_{\mathcal{R}}\delta \le \frac12$. The ball is complete by the identification following (4.5), and the preceding estimate makes it invariant under Φ. The contraction principle gives a fixed point v there, with (4.24). Let γ be the first component of $\mathcal{R}[e^{t\Delta}\chi - F^{\mathrm{a}} + \mathcal{B}(v,v)]$, whose second component is v. Then (γ, v) satisfies (4.17), by (4.22), and the last assertion follows from the equivalence stated before (4.17). □

5. Blow-up on $\mathbb{R}^3$

We prove Proposition 2.1 by applying the right inverse of Proposition 4.3 and the contraction of Lemma 4.8 to every sufficiently close initial velocity. To obtain compactly supported data, we then take for w_0 the solenoidal cutoff of $u^*(0)$ constructed below. The resulting datum is $w_0 + \gamma$, with γ in the fixed space E. The cutoff radius controls the gradient error, while the L^2 norm of the datum is allowed to depend on that radius. Section 6 uses the same cutoff at every time.

5.1. The cutoff. Multiplication of $u^*(t)$ by a spatial cutoff would introduce the divergence $r^{-1}(\nabla\eta)(x/r)\cdot u^*(t,x)$. We cut off a vector potential and take its curl, which keeps the velocity divergence-free. The extra term is supported in the transition annulus, where (4.1) bounds its gradient. We fix a function $\eta \in C^\infty(\mathbb{R}^3)$ with $0 \le \eta \le 1$, $\eta = 1$ on $B_{1/2}$ and $\eta = 0$ outside B_1, and for $r > 0$ and $0 \le t < 1$ we let

$$A(t,x) = \int_0^1 \sigma\, u^*(t,\sigma x) \times x\, d\sigma, \qquad \bar u(t) = \operatorname{curl}\big(\eta(\cdot/r)\, A(t)\big). \tag{5.1}$$

We do not show the radius r in the notation $\bar u$; it is fixed wherever the field is used.

Lemma 5.1. *There exists a constant C such that for every $r > 0$ the field $\bar u$ of (5.1) is smooth on $[0,1)\times\mathbb{R}^3$, and for $0 \le t < 1$ the field $\bar u(t)$ belongs to $C^\infty_{c,\mathrm{div}}$, equals $u^*(t)$ on $B_{r/2}$, vanishes outside B_r, and satisfies*

$$\|\bar u(t) - u^*(t)\|_{\dot H^1} \le C\, r^{-1/2}, \qquad \|\bar u(t)\|_{L^2(\mathbb{R}^3)} \le C\, r^{1/2}. \tag{5.2}$$

Proof. Fix $r > 0$ and $0 \le t < 1$; all constants below are independent of r and t. Differentiation under the integral over $[0,1]$, locally in (t,x) with $t < 1$, gives the smoothness of A and $\bar u$. Since $u^*(t)$ is divergence-free,

$$\operatorname{curl}_x[\sigma\, u^*(t,\sigma x)\times x] = \partial_\sigma[\sigma^2\, u^*(t,\sigma x)],$$

and integration in σ gives $\operatorname{curl} A(t) = u^*(t)$: the term at $\sigma = 0$ vanishes because of the factor σ^2. The product rule gives

$$\bar u(t) = \eta(\cdot/r)\, u^*(t) + r^{-1}(\nabla\eta)(\cdot/r)\times A(t).$$

Thus $\bar u(t)$, which is divergence-free as a curl, equals $u^*(t)$ on $B_{r/2}$, where $\eta(\cdot/r) = 1$, and vanishes outside B_r; in particular it belongs to $C^\infty_{c,\mathrm{div}}$.

For $x \neq 0$, (4.1) bounds the integrand of A by C_W. Its derivative in a unit spatial direction has terms bounded by $\sigma^2|\nabla u^*(t,\sigma x)|\,|x|$ and $\sigma|u^*(t,\sigma x)|$. Each is at most $C_W/|x|$ for $\sigma > 0$; smoothness covers the endpoint. Thus

$$|A(t,x)| \le C_W, \qquad |\nabla A(t,x)| \lesssim |x|^{-1} \qquad (x \neq 0).$$

The field $\bar{u}(t) - u^*(t) = (\eta(\cdot/r) - 1)\,u^*(t) + r^{-1}(\nabla\eta)(\cdot/r) \times A(t)$ vanishes on $B_{r/2}$ and equals $-u^*(t)$ outside B_r. On $r/2 \le |x| \le r$, the product rule gives terms controlled by $|\nabla u^*|$, $r^{-1}|u^*|$, $r^{-1}|\nabla A|$ and $r^{-2}|A|$, each of order r^{-2}. Outside B_r the gradient is bounded by $C_W|x|^{-2}$. Hence

$$\|\bar{u}(t) - u^*(t)\|_{\dot{H}^1}^2 \lesssim r^{-4}r^3 + \int_{|x|\ge r} \frac{dx}{|x|^4} \lesssim r^{-1}.$$

By the same bounds $|\bar{u}(t,x)| \lesssim |x|^{-1}$ for $|x| \le r$, so that $\|\bar{u}(t)\|_{L^2}^2 \lesssim \int_{|x|\le r} |x|^{-2}\,dx \lesssim r$. This is (5.2). □

5.2. Proof of Proposition 2.1. Take $0 < \kappa < 1/4$, E, $C_{\mathcal{R}}$ and $\mathcal{R}$ from Proposition 4.3, $c_* > 0$ from Lemma 4.2, and ε_1 from Lemma 4.8 with $\mathcal{X} = X$ for this κ. Choose $\varepsilon_0 > 0$ so that

$$C_{\mathcal{R}}^2\,\varepsilon_0 \le \varepsilon_1, \qquad 4\,C_{\mathcal{R}}\,\varepsilon_0 \le c_*.$$

For $w_0 \in X$ within ε_0 of $W(0)$ in $\dot{H}^1$, put $\chi = w_0 - u^*(0)$. By (1.4) and (4.20), $\|e^{t\Delta}\chi\|_\kappa \le \|\chi\|_{\dot{H}^1} \le \varepsilon_0$. Since u^* solves the equation (Proposition 4.1), only the heat evolution of χ enters (4.23). The first condition on ε_0 permits the contraction of Lemma 4.8 with $u^{\mathrm{a}} = u^*$, $J : E \hookrightarrow X$ and right inverse (4.8). It gives $\gamma \in E$ and $v \in C_\kappa([0,1); X)$ with $u = u^* + v$ a solution on $[0,1)$ and $u(0) = w_0 + \gamma$. By (4.24) and the second condition on ε_0,

$$\|u(t) - u^*(t)\|_{\dot{H}^1} = \|v(t)\|_{\dot{H}^1} \le 2\,C_{\mathcal{R}}\,\varepsilon_0(1-t)^{\kappa-\frac{1}{4}} \le \tfrac{1}{2}\,c_*(1-t)^{\kappa-\frac{1}{4}} \qquad (0 \le t < 1).$$

Since $(1-t)^\kappa \le 1$, subtracting this error from the lower bound in (4.4) leaves the coefficient $c_*/2$. The triangle inequality gives the upper bound, proving (2.5) with $c = c_*/2$ and C independent of w_0.

Lipschitz selection of the correction. Fix $\mathcal{R}$ from (4.8). For two data w_0, w_0' in this ball denote the selected pairs by $(\gamma, v), (\gamma', v')$. Both trajectories lie in the ball of radius $2C_{\mathcal{R}}\varepsilon_0$ on which the quadratic part of the map in Lemma 4.8 has Lipschitz constant at most $1/2$. Subtract the inputs to $\mathcal{R}$ at the two fixed points. By (4.8), (4.19) and (4.20),

$$\max\{\|\gamma - \gamma'\|_{\dot{H}^1}, \|v - v'\|_\kappa\} \le C_{\mathcal{R}}\|w_0 - w_0'\|_{\dot{H}^1} + \tfrac{1}{2}\|v - v'\|_\kappa.$$

Absorbing the last term bounds the maximum by $2C_{\mathcal{R}}\|w_0 - w_0'\|_{\dot{H}^1}$. At $w_0' = W(0)$ the fixed point is zero and $\gamma' = 0$. Thus $\|\gamma\|_{\dot{H}^1} \le 2C_{\mathcal{R}}\|w_0 - W(0)\|_{\dot{H}^1}$.

Choose r so large that $Cr^{-1/2} \le \varepsilon_0$, with C from Lemma 5.1. Then $w_0 = \bar{u}(0)$ is smooth, solenoidal and compactly supported, and (5.2) puts it within ε_0 of $W(0)$. The preceding construction gives $u(0) = \bar{u}(0) + \gamma$. Both summands belong to $C^\infty_{c,\mathrm{div}}$. □

6. Blow-up on the torus

We prove Proposition 2.2. Retain $c_* > 0$ from (4.4) and $0 < \kappa < 1/4$, E and $\mathcal{R}$ from Proposition 4.3. The torus has period $L \ge 1$. As in Subsection 3.6, $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ consists of mean-zero solenoidal fields with norm $\|u\|_{\dot{H}^1(\mathbb{T}^3_L)} = \|\nabla u\|_{L^2(Q_L)}$ as in (3.15).

We look for a solution in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ on $[0,1)$ of the following form, where Π_L denotes L-periodization (Subsection 6.1). We take L large enough that every $\gamma \in E$ fits inside the support ball $B_{L/8}$ (Subsection 6.3).

$$u = u^{\mathrm{a}} + z, \qquad u(0) = u^{\mathrm{a}}(0) + \Pi_L\gamma,$$

Here $\gamma \in E$ and $z \in C_\kappa([0,1); \dot H^1_{\mathrm{div}}(\mathbb{T}^3_L))$. The field u^{a} is the periodization of the curl of the cutoff potential at radius $L/8$. On $B_{L/16}$ it agrees with u^*. Its residual in the periodic integral equation has weighted norm $O(L^{-1/2})$ (Lemma 6.2). There is also a linear error. When we use the whole-space inverse to solve the periodic linearized equation with right-hand side g, the defect has norm $O(L^{-1/2})\|g\|_\kappa$ (Lemma 6.3). For large L the Neumann series removes the linear defect. With this exact right inverse, Lemma 4.8 removes the nonlinear residual.

Throughout this section, constants in $\lesssim$ depend only on the fixed profile, its right inverse and the cutoff η of Subsection 5.1. They are uniform in L and t, so increasing the period makes the errors in Lemmas 6.2–6.3 small simultaneously.

6.1. Comparison of the heat flows on $\mathbb{T}^3_L$ and $\mathbb{R}^3$. For a field f on $\mathbb{R}^3$ that vanishes almost everywhere outside $\bar B_{L/8}$ let

$$\Pi_L f = \sum_{j \in \mathbb{Z}^3} f(\cdot - Lj)$$

be its periodization; we use the same notation for matrix fields. The translates have disjoint supports, and only the central copy meets Q_L. Thus $\Pi_L f = f$ on Q_L and $\|\Pi_L f\|_{L^p(Q_L)} = \|f\|_{L^p(\mathbb{R}^3)}$. For $f \in L^1$, unfolding the integral over Q_L gives the Fourier coefficients $L^{-3}\hat f(k)$, $k \in L^{-1}\mathbb{Z}^3$, with the whole-space Fourier convention of Subsection 1.4 and the periodic normalization of Subsection 3.6. Let $f \in C^\infty_{c,\mathrm{div}}$ vanish outside $\bar B_{L/8}$. Then f is the divergence of the matrix field $x \otimes f$ with the entries $x_j f_m$, because $\sum_m \partial_m(x_j f_m) = f_j + x_j \operatorname{div} f$, and integrating this compactly supported divergence gives $\int_{\mathbb{R}^3} f\,dx = 0$. Thus $\Pi_L f$, which is smooth, L-periodic and divergence-free, has mean zero, and

$$\Pi_L f \in \dot H^1_{\mathrm{div}}(\mathbb{T}^3_L), \qquad \|\Pi_L f\|_{\dot H^1(\mathbb{T}^3_L)} = \|f\|_{\dot H^1}. \tag{6.1}$$

Let $F \in W^{1,1}(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ be a matrix field that vanishes almost everywhere outside $\bar B_{L/8}$, and let $\operatorname{div} F$ be its divergence as in Subsection 3.2. The L-periodic field $\operatorname{div} \Pi_L F = \Pi_L \operatorname{div} F$ is integrable on Q_L, and we compare the periodic field $e^{t\Delta}\mathbb{P}\operatorname{div}\Pi_L F$ of Subsection 3.6 with the field $e^{t\Delta}\mathbb{P}\operatorname{div} F$ of (2.2). The Leray projection is nonlocal. Even where the two input fields agree, the projected heat flows feel the other periodic copies. In the differences below, a heat flow or projection applied to a periodized input acts on $\mathbb{T}^3_L$. Its counterpart without Π_L acts on $\mathbb{R}^3$. We compare the outputs on Q_L.

Lemma 6.1. *There exists a constant C such that for every $L \ge 1$, every $t > 0$ and every $F \in W^{1,1}(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ that vanishes almost everywhere outside $\bar B_{L/8}$, the field $g = e^{t\Delta}\mathbb{P}\operatorname{div}\Pi_L F - e^{t\Delta}\mathbb{P}\operatorname{div} F$ satisfies*

$$\|\nabla g\|_{L^2(Q_L)} + \|g\|_{L^6(Q_L)} \le C\, L^{-7/2}\|F\|_{L^1(\mathbb{R}^3)}. \tag{6.2}$$

Proof. In this proof the constants in $\lesssim$ are absolute. Let $h = \nabla O_t * F$, so that $h = e^{t\Delta}\mathbb{P}\operatorname{div} F$ by (3.10). By (3.9) with $l = 1$ and $l = 2$, the first and second derivatives of O_t are bounded and integrable, and

$$|\nabla O_t(w)| \lesssim |w|^{-4}, \qquad |\nabla^2 O_t(w)| \lesssim |w|^{-5} \qquad (w \ne 0,\ t > 0).$$

In particular h is integrable and continuously differentiable, with $\partial_i h = \partial_i \nabla O_t * F$.

Let $x \in \mathbb{R}^3$ with $\max_i |x_i| < \frac{L}{2}$, let $x' \in \bar B_{L/8}$ and $j \in \mathbb{Z}^3 \setminus \{0\}$. With $|j|_\infty = \max_i |j_i|$ we have $|x - x' - Lj| \ge L\,|j|_\infty - \frac{5}{8}L \ge \frac{3}{8}L\,|j|_\infty$, and since F vanishes almost everywhere outside $\bar B_{L/8}$, we obtain

$$|h(x - Lj)| \lesssim (L\,|j|_\infty)^{-4}\|F\|_{L^1}, \qquad |\nabla h(x - Lj)| \lesssim (L\,|j|_\infty)^{-5}\|F\|_{L^1}.$$

There are $O(n^2)$ lattice points with $|j|_\infty = n$. The shells therefore give summable bounds $\lesssim L^{-4}n^{-2}\|F\|_{L^1}$ and $\lesssim L^{-5}n^{-3}\|F\|_{L^1}$. Thus $g_0 = \sum_{j\neq 0} h(\cdot - Lj)$ and its first derivatives converge uniformly on $\max_i |x_i| < \frac{L}{2}$, with

$$|g_0| \lesssim L^{-4}\|F\|_{L^1}, \qquad |\nabla g_0| \lesssim L^{-5}\|F\|_{L^1}.$$

To identify g_0, compare Fourier coefficients. Since $h \in L^1$, its periodization has coefficients $L^{-3}\hat{h}(k)$, where $\hat{h}(k) = e^{-4\pi^2 t|k|^2}\,\mathbb{P}_k\,\widehat{\operatorname{div} F}(k)$ by (2.2) and (3.10). Periodization gives $\operatorname{div}\Pi_L F$ the coefficients $L^{-3}\widehat{\operatorname{div} F}(k)$, so Subsection 3.6 gives precisely the same coefficients for its projected heat flow. Fourier uniqueness therefore gives $e^{t\Delta}\mathbb{P}\operatorname{div}\Pi_L F = h + g_0$, that is, $g = g_0$ and $\nabla g = \nabla g_0$, almost everywhere on the cube. Since the cube has volume L^3, the bounds for g_0 give

$$\|g\|_{L^6(Q_L)} \lesssim L^{1/2}L^{-4}\|F\|_{L^1}, \qquad \|\nabla g\|_{L^2(Q_L)} \lesssim L^{3/2}L^{-5}\|F\|_{L^1},$$

which is (6.2). □

For $f \in C^\infty_{c,\mathrm{div}}$ supported in $\bar{B}_{L/8}$, take $F = x \otimes f$. Then $f = \operatorname{div} F$ and $|F| \leq \frac{L}{8}|f|$. Since $\mathbb{P}_\xi \hat{f}(\xi) = \hat{f}(\xi)$ for every ξ, the projected heat flows are $e^{t\Delta}f$ in X and $e^{t\Delta}\Pi_L f$ in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$. Thus (6.2) gives

$$\|\nabla(e^{t\Delta}\Pi_L f - e^{t\Delta}f)\|_{L^2(Q_L)} + \|e^{t\Delta}\Pi_L f - e^{t\Delta}f\|_{L^6(Q_L)} \lesssim L^{-5/2}\|f\|_{L^1(\mathbb{R}^3)} \qquad (t>0). \tag{6.3}$$

6.2. The approximate solution. Let $\bar{u}$ be the field (5.1) for the radius $r = L/8$. By Lemma 5.1, $\bar{u}$ is smooth on $[0,1)\times\mathbb{R}^3$, and the field $\bar{u}(t)$ belongs to $C^\infty_{c,\mathrm{div}}$, equals $u^*(t)$ on $B_{L/16}$, vanishes outside $\bar{B}_{L/8}$, and satisfies

$$\|\bar{u}(t) - u^*(t)\|_{\dot{H}^1} \lesssim L^{-1/2}, \qquad \|\bar{u}(t)\|_{L^2(\mathbb{R}^3)} \lesssim L^{1/2} \qquad (0 \leq t < 1). \tag{6.4}$$

On each compact time interval in $[0,1)$, smoothness and the common compact support give continuity of $t \mapsto \nabla\bar{u}(t)$ in L^2, hence of $t \mapsto \bar{u}(t)$ in X. The approximate solution is the periodization of this field,

$$u^{\mathrm{a}}(t) = \Pi_L \bar{u}(t) \qquad (0 \leq t < 1).$$

By (6.1), $u^{\mathrm{a}} : [0,1) \to \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ is continuous. Let F^{a} be its residual (4.16), and let $\mathcal{B}_{\mathbb{T}}$ denote (4.6) with K replaced by $K_{\mathbb{T}}$.

Lemma 6.2. *There exists a constant C such that for every $L \geq 1$ the maps u^{a} and F^{a} satisfy*

$$(c_* - C\,L^{-1/2})(1-t)^{-1/4} \leq \|u^{\mathrm{a}}(t)\|_{\dot{H}^1(\mathbb{T}^3_L)} = \|\bar{u}(t)\|_{\dot{H}^1} \leq C\,(1-t)^{-1/4} \qquad (0 \leq t < 1), \tag{6.5}$$

and $F^{\mathrm{a}} \in C_\kappa([0,1); \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L))$ with $\|F^{\mathrm{a}}\|_\kappa \leq C\,L^{-1/2}$.

Proof. The equality in (6.5) is (6.1), and the two bounds follow from the bounds (4.4) for u^* and from the first bound of (6.4), because $L^{-1/2} \leq 1 \leq (1-t)^{-1/4}$. In particular $\|\bar{u}\|_0 \lesssim 1$ and $\|u^*\|_0 \lesssim 1$. We next bound the residual of $\bar{u}$ on $\mathbb{R}^3$, the map $\bar{F} = \bar{u} - e^{t\Delta}\bar{u}(0) - \mathcal{B}(\bar{u},\bar{u})$ from $[0,1)$ to X. Since $u^* = e^{t\Delta}u^*(0) + \mathcal{B}(u^*,u^*)$ and $\mathcal{B}$ is bilinear,

$$\bar{F} = (\bar{u} - u^*) - e^{t\Delta}\big(\bar{u}(0) - u^*(0)\big) - \mathcal{B}(\bar{u}-u^*,\bar{u}) - \mathcal{B}(u^*,\bar{u}-u^*).$$

By (6.4), $\bar{u} - u^*$ is continuous and bounded by $CL^{-1/2}$ in X. Since $0 < \kappa < 1/4$, the weight $(1-t)^{1/4-\kappa}$ in (4.5) is at most one. Estimate the four terms by (6.4), (4.20), and twice (4.21), using the symmetry of $\mathcal{B}$ and $\bar{u}, u^* \in C_0([0,1);X)$. This gives

$$\|\bar{F}\|_\kappa \lesssim L^{-1/2}(1 + \|\bar{u}\|_0 + \|u^*\|_0) \lesssim L^{-1/2}.$$

It remains to pass to the torus. Since $\bar u(t')$ is divergence-free, $(\bar u(t')\cdot\nabla)\bar u(t') = \operatorname{div}\big(\bar u(t')\otimes\bar u(t')\big)$, and $(\Pi_L\bar u\cdot\nabla)\Pi_L\bar u = \Pi_L[(\bar u\cdot\nabla)\bar u]$. Hence the definitions (2.3) and (3.17) give

$$\bar F(t) = \bar u(t) - e^{t\Delta}\bar u(0) + \int_0^t e^{(t-t')\Delta}\mathbb{P}\operatorname{div}\big(\bar u(t')\otimes\bar u(t')\big)\,dt',$$

$$F^{\mathrm a}(t) = \Pi_L\bar u(t) - e^{t\Delta}\Pi_L\bar u(0) + \int_0^t e^{(t-t')\Delta}\mathbb{P}\operatorname{div}\Pi_L\big(\bar u(t')\otimes\bar u(t')\big)\,dt'.$$

Here $\bar F$ is a whole-space field and $F^{\mathrm a}$ is periodic. Both vanish at $t=0$. For $0<t<1$ we restrict each to Q_L, where $\Pi_L\bar u(t)=\bar u(t)$. The field $\bar u(0)$ and the matrix fields $\bar u(t')\otimes\bar u(t')$ are smooth and vanish outside $\bar B_{L/8}$. We apply (6.3) to the second terms, and Lemma 6.1 to the third terms under the integrals:

$$\|\nabla(F^{\mathrm a}(t)-\bar F(t))\|_{L^2(Q_L)} \lesssim L^{-5/2}\|\bar u(0)\|_{L^1} + L^{-7/2}\int_0^t \|\bar u(t')\|_{L^2}^2\,dt' \lesssim L^{-1/2}.$$

Indeed, support and (6.4) give $\|\bar u(0)\|_1 \le |B_{L/8}|^{1/2}\|\bar u(0)\|_2 \lesssim L^2$ and $\|\bar u(t')\otimes\bar u(t')\|_1 = \|\bar u(t')\|_2^2 \lesssim L$. The two contributions are therefore $O(L^{-1/2})$ and $O(L^{-5/2})$. Since $\|F^{\mathrm a}(t)\|_{\dot H^1(\mathbb{T}^3_L)} = \|\nabla F^{\mathrm a}(t)\|_{L^2(Q_L)}$,

$$\|F^{\mathrm a}(t)\|_{\dot H^1(\mathbb{T}^3_L)} \le \|\bar F(t)\|_{\dot H^1} + \|\nabla(F^{\mathrm a}(t)-\bar F(t))\|_{L^2(Q_L)} \lesssim (1-t)^{\kappa-\frac14}L^{-1/2} + L^{-1/2},$$

and multiplication by $(1-t)^{1/4-\kappa}\le 1$ and the supremum in (4.5) give $\|F^{\mathrm a}\|_\kappa \lesssim L^{-1/2}$. Lemmas 3.7 and 3.3 give its continuity with values in $\dot H^1_{\operatorname{div}}(\mathbb{T}^3_L)$. □

6.3. An approximate right inverse. The product of the periodic disturbance with the cutoff background is supported in one ball of the period cell. This product can act as a whole-space source. The correction returned by the whole-space inverse then enters the periodic heat flow in (6.12). The torus correction to the initial datum is obtained by periodizing the fields of E. Choose $\bar B_{r_E}$ containing the supports of a basis of E. Every $\gamma\in E$ is supported there, and Hölder's inequality and (3.1) give $\|\gamma\|_{L^1}\lesssim\|\gamma\|_{\dot H^1}$, with r_E fixed. Henceforth $L\ge\max\{1,8r_E\}$. By (6.1), $J: E\to\dot H^1_{\operatorname{div}}(\mathbb{T}^3_L)$, $J\gamma=\Pi_L\gamma$, is well-defined and isometric. For $u^{\mathrm a}$, E and J, the operator (4.16) is

$$\mathcal{L}(\gamma,z) = z - 2\mathcal{B}_{\mathbb{T}}(u^{\mathrm a},z) - e^{t\Delta}\Pi_L\gamma. \tag{6.6}$$

From the right inverse $\mathcal{R}$ on $\mathbb{R}^3$ we construct a linear map $g\mapsto(\gamma,z)$ with $\mathcal{L}(\gamma,z)=g$ up to an error of order $L^{-1/2}$.

Let $B=B_{L/8}$. Use $\|v\|_B=\|\nabla v\|_{L^2(B)}+\|v\|_{L^6(B)}$ on $H^1(B)$ and in $C_\kappa([0,1);H^1(B))$. The equivalence with the usual $H^1(B)$ norm can depend on L. The following restriction bounds in $\|\cdot\|_B$ are uniform in L. For either ambient space, $v|_B$ denotes restriction to B, applied at each time for a trajectory. Since $B\subset Q_L$, (3.1) and (3.18) give

$$\|v|_B\|_B\lesssim\|v\|_{\dot H^1}\quad(v\in X),\qquad \|v|_B\|_B\lesssim\|v\|_{\dot H^1(\mathbb{T}^3_L)}\quad(v\in\dot H^1_{\operatorname{div}}(\mathbb{T}^3_L)). \tag{6.7}$$

Thus the restriction to B of a continuous map from $[0,1)$ to one of these spaces is a continuous map from $[0,1)$ to $H^1(B)$, and its values are divergence-free.

Let $\zeta:[0,1)\to H^1(B)$ be continuous and divergence-free. Extend the tensor $\zeta(t)\otimes\bar u(t)+\bar u(t)\otimes\zeta(t)$ by zero and call the result $F_\zeta(t)$. Taking any $H^1(\mathbb{R}^3)$ extension of $\zeta(t)$ gives the same product, because $\bar u(t)$ is smooth and supported in $\bar B$. Thus $F_\zeta(t)\in W^{1,1}(\mathbb{R}^3;\mathbb{R}^{3\times3})$. Its divergence is the zero extension $f_\zeta(t)$ of $(\bar u(t)\cdot\nabla)\zeta(t)+(\zeta(t)\cdot\nabla)\bar u(t)$, since both fields have

zero divergence on B. Integrating this compactly supported divergence gives $\int_{\mathbb{R}^3} f_\zeta(t) = 0$. Hölder's inequality, (3.1) and (6.5) give

$$\|f_\zeta(t)\|_{L^{3/2}} \le \|\bar u(t)\|_{L^6}\|\nabla\zeta(t)\|_{L^2(B)} + \|\zeta(t)\|_{L^6(B)}\|\nabla\bar u(t)\|_{L^2} \lesssim (1-t)^{-1/4}\|\zeta(t)\|_B, \tag{6.8}$$

and f_ζ is continuous from $[0,1)$ to $L^{3/2}(\mathbb{R}^3)$. Both operators below take a trajectory on B. The first evolves its interaction with $\bar u$ in the whole space, the second periodizes the same interaction and evolves it on $\mathbb{T}^3_L$. Let

$$(\mathcal{A}\zeta)(t) = -\int_0^t e^{(t-t')\Delta}\mathbb{P}\, f_\zeta(t')\,dt', \qquad (\mathcal{A}_{\mathbb{T}}\zeta)(t) = -\int_0^t e^{(t-t')\Delta}\mathbb{P}\,\Pi_L f_\zeta(t')\,dt'.$$

By Lemmas 3.1 and 3.7 for $0 \le t' < t$ the two integrands are continuous in t' and in t, with values in X and in $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$, the latter because $\Pi_L f_\zeta(t')$ has mean zero, and with norms at most a constant multiple of $(t-t')^{-3/4}\|f_\zeta(t')\|_{L^{3/2}}$. For fixed $t<1$, $\|f_\zeta(t')\|_{L^{3/2}}$ is bounded on $[0,t]$ and $(t-t')^{-3/4}$ is integrable. Hence the integrals converge and are continuous in t, as in the proof of Lemma 3.3 (Appendix A.1). Since $\bar u(t)$ vanishes outside $\bar B$, the definitions (2.3) and (3.17) give, for continuous $v:[0,1)\to X$ and $z:[0,1)\to \dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$,

$$2\mathcal{B}(\bar u, v) = \mathcal{A}(v|_B), \qquad 2\mathcal{B}_{\mathbb{T}}(u^{\mathrm{a}}, z) = \mathcal{A}_{\mathbb{T}}(z|_B). \tag{6.9}$$

If moreover $\zeta \in C_\kappa([0,1); H^1(B))$, then (6.8) and (4.18) give

$$\|(\mathcal{A}\zeta)(t)\|_{\dot H^1} \lesssim \|\zeta\|_\kappa \int_0^t (t-t')^{-3/4}(1-t')^{\kappa-\frac12}\,dt' \lesssim (1-t)^{\kappa-\frac14}\|\zeta\|_\kappa,$$

and Lemma 3.7 gives this bound for $(\mathcal{A}_{\mathbb{T}}\zeta)(t)$ in $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$. Thus

$$\|\mathcal{A}\zeta\|_\kappa \lesssim \|\zeta\|_\kappa, \qquad \|\mathcal{A}_{\mathbb{T}}\zeta\|_\kappa \lesssim \|\zeta\|_\kappa. \tag{6.10}$$

Moreover, on the ball B the two integrals are close to each other. By (6.4) and by Hölder's inequality on B, whose volume is at most L^3,

$$\|F_\zeta(t)\|_{L^1} \le 2\,\|\bar u(t)\|_{L^2}\,\|\zeta(t)\|_{L^2(B)} \lesssim L^{1/2} L\,\|\zeta(t)\|_{L^6(B)} \le L^{3/2}\|\zeta(t)\|_B,$$

and (6.2) under the integrals gives, since $B \subset Q_L$,

$$\|((\mathcal{A}_{\mathbb{T}}\zeta)(t) - (\mathcal{A}\zeta)(t))|_B\|_B \lesssim L^{-7/2}\int_0^t \|F_\zeta(t')\|_{L^1}\,dt' \lesssim L^{-2}\|\zeta\|_\kappa \int_0^1 (1-t')^{\kappa-\frac14}\,dt' \lesssim L^{-2}\|\zeta\|_\kappa.$$

Multiplying by $(1-t)^{1/4-\kappa} \le 1$ gives

$$\|(\mathcal{A}_{\mathbb{T}}\zeta - \mathcal{A}\zeta)|_B\|_\kappa \lesssim L^{-2}\|\zeta\|_\kappa. \tag{6.11}$$

Take $g \in C_\kappa([0,1); \dot H^1_{\mathrm{div}}(\mathbb{T}^3_L))$. To construct a right inverse, first write $z = g + h$ in the exact torus equation $\mathcal{L}(\gamma, z) = g$. By (6.6) and (6.9) the equation for h is

$$h - \mathcal{A}_{\mathbb{T}}(h|_B) - e^{t\Delta}\Pi_L\gamma = \mathcal{A}_{\mathbb{T}}(g|_B).$$

Only the interaction with the background remains on the right. Use its whole-space counterpart $\mathcal{A}(g|_B)$ as input to $\mathcal{R}$. Equations (6.7) and (6.10) place it in $C_\kappa([0,1); X)$. If (γ, v) is the output of $\mathcal{R}$, the restrictions of g and v belong to the same space on B. Their sum determines the periodic disturbance. Define $\mathcal{R}^{\mathrm{a}} g = (\gamma, z)$ by

$$(\gamma, v) = \mathcal{R}(\mathcal{A}(g|_B)), \qquad \zeta = (g+v)|_B, \qquad z = g + \mathcal{A}_{\mathbb{T}}\zeta + e^{t\Delta}\Pi_L\gamma. \tag{6.12}$$

The defect now depends only on how well ζ predicts the restriction of z to B. By (6.9) and the definition of z,

$$\mathcal{L}(\gamma, z) - g = \mathcal{A}_{\mathbb{T}}(\zeta - z|_B), \tag{6.13}$$

where both terms on the right are defined on B before applying $\mathcal{A}_{\mathbb{T}}$. The whole-space equation (4.8) for (γ, v) reads $v = \mathcal{A}(g|_B) + 2\mathcal{B}(u^*, v) + e^{t\Delta}\gamma$, and $2\mathcal{B}(u^*, v) = \mathcal{A}(v|_B) - 2\mathcal{B}(\bar{u} - u^*, v)$, by (6.9), so that

$$v = \mathcal{A}\zeta + e^{t\Delta}\gamma - 2\mathcal{B}(\bar{u} - u^*, v). \tag{6.14}$$

Lemma 6.3. *There exists a constant C such that for every $L \geq \max\{1, 8\, r_E\}$ the map $g \mapsto (\gamma, z)$ defined by (6.12) is linear from $C_\kappa([0,1); \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L))$ to $E \times C_\kappa([0,1); \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L))$, and*

$$\max\{\|\gamma\|_{\dot{H}^1}, \|z\|_\kappa\} \leq C\, \|g\|_\kappa, \qquad \|\mathcal{L}(\gamma, z) - g\|_\kappa \leq C\, L^{-1/2} \|g\|_\kappa, \tag{6.15}$$

where $\mathcal{L}$ is the operator (6.6).

Proof. Linearity follows from that of $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{A}_{\mathbb{T}}$. For g, γ, v, ζ and z as in (6.12), (6.7) and (6.10) put $\mathcal{A}(g|_B)$ in $C_\kappa([0,1); X)$ with norm $\lesssim \|g\|_\kappa$. Proposition 4.3 and restriction by (6.7) give

$$\max\{\|\gamma\|_{\dot{H}^1}, \|v\|_\kappa\} \lesssim \|g\|_\kappa, \qquad \|\zeta\|_\kappa \lesssim \|g\|_\kappa,$$

and by (6.10), (4.20) and (6.1),

$$\|z\|_\kappa \leq \|g\|_\kappa + \|\mathcal{A}_{\mathbb{T}}\zeta\|_\kappa + \|\Pi_L \gamma\|_{\dot{H}^1(\mathbb{T}^3_L)} \lesssim \|g\|_\kappa.$$

This is the first bound of (6.15).

By (6.13) and (6.10) it remains to bound $\zeta - z|_B$. Since $\zeta - z|_B = (v - \mathcal{A}_{\mathbb{T}}\zeta - e^{t\Delta}\Pi_L\gamma)|_B$, by the definitions of ζ and z, the identity (6.14) gives

$$\zeta - z|_B = (\mathcal{A}\zeta - \mathcal{A}_{\mathbb{T}}\zeta)|_B + (e^{t\Delta}\gamma - e^{t\Delta}\Pi_L\gamma)|_B - 2\mathcal{B}(\bar{u} - u^*, v)|_B.$$

By (6.11) the first term has norm at most a constant multiple of $L^{-2}\|\zeta\|_\kappa$ in $C_\kappa([0,1); H^1(B))$. By (6.3) for $f = \gamma$, the second has norm $\|\cdot\|_B$ at most a constant multiple of $L^{-5/2}\|\gamma\|_{L^1} \lesssim L^{-5/2}\|\gamma\|_{\dot{H}^1}$ at every time $t \in (0,1)$, and it vanishes at $t = 0$. By (6.7), by (4.21) with the bounded map $\bar{u} - u^*$ and by (6.4), the third has norm at most a constant multiple of $L^{-1/2}\|v\|_0 \leq L^{-1/2}\|v\|_\kappa$. The comparison of the two projected interaction terms costs L^{-2}, and that of the two heat evolutions of γ costs $L^{-5/2}$. Replacing u^* by its cutoff costs $L^{-1/2}$ and determines the order of the total error. The bounds for γ, v and ζ therefore give

$$\|\zeta - z|_B\|_\kappa \lesssim L^{-1/2}\|g\|_\kappa,$$

which gives the second bound of (6.15). □

6.4. Proof of Proposition 2.2. Let C_1 and C_2 be the constants of Lemmas 6.2 and 6.3, and let ε_1 be the number of Lemma 4.8 for the spaces $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$. It is one number for all L, because it depends only on κ and on the constant of (3.12), which is $C_{\mathbb{T}}$ on each of these spaces (Lemma 3.7). We fix $L \geq \max\{1, 8\, r_E\}$ so large that

$$C_2 L^{-1/2} \leq \tfrac{1}{2}, \qquad 4\, C_1 C_2^2 L^{-1/2} \leq \varepsilon_1, \qquad C_1(1 + 4\, C_2)\, L^{-1/2} \leq \tfrac{1}{2}\, c_*. \tag{6.16}$$

On $C_\kappa([0,1); \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L))$ set $\mathcal{E} = I - \mathcal{L}\mathcal{R}^{\mathrm{a}}$ with $\mathcal{R}^{\mathrm{a}}$ from (6.12). Lemma 6.3 and the first condition of (6.16) give $\|\mathcal{E}\| \leq 1/2$. Hence $(I - \mathcal{E})^{-1} = \sum_{n \geq 0} \mathcal{E}^n$ has norm at most 2. The exact right inverse is

$$\mathcal{R}_{\mathbb{T}} = \mathcal{R}^{\mathrm{a}} \sum_{n \geq 0} (I - \mathcal{L}\mathcal{R}^{\mathrm{a}})^n$$

Indeed $\mathcal{L}\mathcal{R}_{\mathbb{T}} = I$. For $(\gamma, z) = \mathcal{R}_{\mathbb{T}} g$, the first bound of (6.15) gives $\|z\|_\kappa \leq 2C_2\|g\|_\kappa$. This is (4.22) with $C_{\mathcal{R}} = 2C_2$. Lemma 6.2 and the second condition of (6.16) give $(2C_2)^2\|F^{\mathrm{a}}\|_\kappa \leq \varepsilon_1$,

which is (4.23) with $\chi = 0$. Lemma 4.8 gives $\gamma \in E$ and $z \in C_\kappa([0,1); \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L))$ such that $u = u^{\mathrm{a}} + z$ is a solution on $[0,1)$ with

$$u(0) = u^{\mathrm{a}}(0) + \Pi_L \gamma = \Pi_L\big(\bar{u}(0) + \gamma\big), \qquad \|z\|_\kappa \le 4\, C_2\, \|F^{\mathrm{a}}\|_\kappa \le 4\, C_1 C_2 L^{-1/2}, \tag{6.17}$$

the first bound by (4.24).

This solution satisfies

$$\tfrac{1}{2}\, c_*(1-t)^{-1/4} \le \|u(t)\|_{\dot{H}^1(\mathbb{T}^3_L)} \lesssim (1-t)^{-1/4} \qquad (0 \le t < 1). \tag{6.18}$$

Indeed, $\|z(t)\|_{\dot{H}^1(\mathbb{T}^3_L)} \le (1-t)^{\kappa - \frac{1}{4}} \|z\|_\kappa \le (1-t)^{-1/4} \|z\|_\kappa$. Subtract this bound from the lower bound in (6.5) and use (6.17). The coefficient is at least $c_* - C_1(1 + 4C_2)L^{-1/2} \ge c_*/2$ by (6.16). Addition gives the upper bound.

We now pass to the torus $\mathbb{T}^3$. Let

$$T = L^{-2}, \qquad u_{\mathrm{per}}(t) = D_{L^2}\, u(L^2 t) \qquad (0 \le t < T),$$

so $0 < T \le 1$ and $u_{\mathrm{per}}(t,x) = L\, u(L^2 t, Lx)$. By (3.21), D_{L^2} maps $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ onto $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$, multiplying the norm by $L^{1/2}$; hence u_{per} is continuous on $[0,T)$. Apply D_{L^2} to (2.4) at time $L^2 t$ and substitute $t' = L^2 \tau$ in its integral. The last two identities of (3.21) give (2.4) for u_{per} on $[0,T)$, so it is a solution in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$. Its datum $u_0 = D_{L^2} \Pi_L(\bar{u}(0) + \gamma)$ periodizes $D_{L^2}(\bar{u}(0) + \gamma) \in C^\infty_{c,\mathrm{div}}$, supported in $\bar{B}_{1/8}$, with period one. Thus u_0 is smooth, periodic and divergence-free.

It remains to deduce the bounds (2.6). Since

$$\|u_{\mathrm{per}}(t)\|_{\dot{H}^1(\mathbb{T}^3)} = L^{1/2} \|u(L^2 t)\|_{\dot{H}^1(\mathbb{T}^3_L)}, \qquad L^{1/2}(1 - L^2 t)^{-1/4} = (T-t)^{-1/4},$$

(6.18) and the norm equivalence (3.16) give the first bound in (2.6), with $c = \frac{1}{2} c_*$. For the second, (3.5) and the identities $L^2/(1 - L^2 t) = 1/(T-t)$ and $-\log(1 - L^2 t) = \log(T/(T-t))$ give $D_{L^2} u^*(L^2 t) = u^*_T(t)$. On $B_{L/16} \subset Q_L$, $u^{\mathrm{a}} = \bar{u} = u^*$, hence $u - u^* = z$. Since $D_{L^2} f = L f(L\,\cdot)$, $u_{\mathrm{per}}(t) - u^*_T(t)$ agrees with $D_{L^2} z(L^2 t)$ on $B_{1/16} \subset Q_1$. This last field lies in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$, so (3.16) and (3.21) give, for $0 \le t < T$,

$$\begin{aligned} \|u_{\mathrm{per}}(t) - u^*_T(t)\|_{H^1(B_{1/16})} &\le \|D_{L^2}\, z(L^2 t)\|_{H^1(\mathbb{T}^3)} \lesssim L^{1/2} \|z(L^2 t)\|_{\dot{H}^1(\mathbb{T}^3_L)} \\ &\le L^{1/2}(1 - L^2 t)^{\kappa - \frac{1}{4}} \|z\|_\kappa = T^{-\kappa}(T-t)^{\kappa - \frac{1}{4}} \|z\|_\kappa. \end{aligned}$$

By (6.17), $T^{-\kappa}\|z\|_\kappa \le 4 C_1 C_2 T^{1/4 - \kappa}$. This proves the local comparison in (2.6) and completes the proof of Proposition 2.2. □

7. From a certificate to a profile

Sections 4–6 start with an exact profile and change the initial datum of a solution that will approach it. Here we must find the profile itself. The computed field $\bar{W}$ of Appendix C supplies the starting approximation for a family W^{a}_α with candidate periods $S(\alpha)$. A reference field W^{r}_α supplies the linear estimate. If the approximate family satisfies the certificate conditions, one member can be corrected to an exact solution. The estimates must hold for every parameter and every perturbation specified in Subsection 7.3. The conclusion selects one parameter α. At normalized time $\theta = s/S(\alpha)$ the exact velocity is $W(S(\alpha)\theta) = W^{\mathrm{a}}_\alpha(\theta) + v(\theta)$.

Proposition 7.8 first solves the periodic linear equation with added forces and constraints. Proposition 7.10 uses its solution to update the trial parameter and the trial quadratic term together. At a fixed point the quadratic term agrees with the one in the equation. The bounds on the parameter update then force every auxiliary force coefficient to vanish. Inequality (7.28) keeps the corrected field nonzero, and Proposition 7.11 turns its weak equation into

the smooth profile with pointwise decay required in Theorem 1.1. Appendix B derives the analytic estimates from conditions on represented data. For a discussion of this distinction in computer-assisted proofs, see [58, §1.1, preprint pp. 2–3].

7.1. The equation in rescaled time. Let $S > 0$. A field W of the self-similar variables (s, y) has period S in s if and only if the field $\mathcal{W}(\theta, y) = W(S\theta, y)$ has period 1 in θ, and in the variables (θ, y) the first equation of (1.2) reads

$$\mathcal{T}_S \mathcal{W} + (\mathcal{W} \cdot \nabla)\mathcal{W} + \nabla q = 0, \qquad \mathcal{T}_S = \tfrac{1}{S}\partial_\theta + \tfrac{1}{2} + \tfrac{1}{2}\, y \cdot \nabla - \Delta. \tag{7.1}$$

Throughout this section the time variable is $\theta = s/S$. The fields of the variables (θ, y) that we consider are defined on $\mathbb{R} \times \mathbb{R}^3$ and, with the exception of test fields, have period 1 in θ. We write $F(\theta) = F(\theta, \cdot)$, and we apply $\mathcal{T}_S$ and the derivatives in y in the sense of distributions on $\mathbb{R} \times \mathbb{R}^3$. Let $\mathcal{H}$ be the Hilbert space of the measurable fields $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ of period 1 in θ that are square integrable on $(0, 1) \times \mathbb{R}^3$, with

$$\langle F, G \rangle_{\mathcal{H}} = \int_0^1 \int_{\mathbb{R}^3} F \cdot G \, dy \, d\theta, \qquad \|F\|_{\mathcal{H}} = \langle F, F \rangle_{\mathcal{H}}^{1/2};$$

we use the same notation for fields with values in the 3×3 matrices, with $F \cdot G = \sum_{j,k} F_{jk} G_{jk}$. Let $\mathcal{H}_{\mathrm{div}}$ be the closed subspace of the divergence-free fields in $\mathcal{H}$, and let $\mathbb{P}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{div}}$. This is the Leray projection applied at each time: for almost every θ it applies to $F(\theta)$ the multiplier $\mathbb{P}_\xi$ of (2.2).

The equation (7.1) is tested against divergence-free fields, so that it determines its terms only up to gradients. For F and G in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$ we say that F and G are *equal modulo gradients*, and write $F \simeq G$, if

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} (F - G) \cdot \varphi \, dy \, d\theta = 0 \qquad \text{for every divergence-free } \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3). \tag{7.2}$$

We say that a field $\mathcal{W}$ *solves (7.1) modulo gradients* if $\mathcal{T}_S \mathcal{W} + (\mathcal{W} \cdot \nabla)\mathcal{W} \simeq 0$. For F and G in $\mathcal{H}$ we have $F \simeq G$ if and only if $\mathbb{P}F = \mathbb{P}G$. Indeed, the integral in (7.2) is the inner product in $\mathcal{H}$ of $F - G$ with the field $\sum_{k \in \mathbb{Z}} \varphi(\theta + k, y)$, which belongs to $\mathcal{H}_{\mathrm{div}}$, and the fields of this form are dense in $\mathcal{H}_{\mathrm{div}}$: by a partition of unity in θ, among them are the products

$$a(\theta)\, \psi(y), \qquad a \in C^\infty(\mathbb{R}) \text{ of period } 1, \qquad \psi \in C^\infty_{c,\mathrm{div}}, \tag{7.3}$$

and finite sums of these products are dense in $\mathcal{H}_{\mathrm{div}}$. To obtain (7.3) by periodization, take $\chi \in C_c^\infty(\mathbb{R})$ with $\sum_k \chi(\theta + k) = 1$ and use $\varphi(\theta, y) = \chi(\theta) a(\theta) \psi(y)$. For the spatial density, the heat multiplier (2.1) first approximates a solenoidal L^2 field by solenoidal H^1 fields; Heywood [60, Theorem 8, p. 89] then approximates each of these by $C^\infty_{c,\mathrm{div}}$ fields. Smooth periodic functions are dense in the time L^2 space, which gives the asserted product density.

The perturbations of the approximate profile lie in the following set. Here $X = \dot{H}^1_{\mathrm{div}}(\mathbb{R}^3)$, with norm $\|u\|_X = \|\nabla u\|_{L^2}$. For $S > 0$ let $\mathcal{D}_S$ be the set of the fields $v \in L^2_{\mathrm{loc}}(\mathbb{R}; X)$ of period 1 in θ such that

$$\Delta v \in \mathcal{H} \qquad \text{and} \qquad \mathcal{T}_S v \in \mathcal{H}. \tag{7.4}$$

The second condition says that $\frac{1}{S}\partial_\theta v + \frac{1}{2} v + \frac{1}{2}(y \cdot \nabla) v$ is square integrable over a period. The domain imposes no separate $\mathcal{H}$ bound on $\partial_\theta v$ or $(y \cdot \nabla) v$. The logarithmic phase in Subsection C.1 gives an example. For $|y| > 1$ and a smooth S-periodic F, the scalar field $f(s, y) = |y|^{-1} F(2 \log |y| - s)$ satisfies $(\partial_s + \frac{1}{2} y \cdot \nabla + \frac{1}{2}) f = 0$. Its separate time and drift derivatives can have order $|y|^{-1}$, while $\Delta f = O(|y|^{-3})$. In (7.4) the same derivative combination occurs with $s = S\theta$. By Lemma 7.4 below, $\mathcal{T}_S$ maps $\mathcal{D}_S$ into $\mathcal{H}_{\mathrm{div}}$, and every $v \in \mathcal{D}_S$ is, after

a modification on a set of times of measure zero, a continuous map from $\mathbb{R}$ to X; we always take this representative.

A certificate may have symmetries. Let Γ be a set of pairs (R, θ_0) of an isometry $R \in O(3)$ and a number $\theta_0 \in \mathbb{R}$; it may be empty. A field F on $\mathbb{R} \times \mathbb{R}^3$ with values in $\mathbb{R}^3$ is Γ-*invariant* if

$$F(\theta + \theta_0, y) = R\, F(\theta, R^{-1}y) \qquad \text{for almost every } (\theta, y) \in \mathbb{R} \times \mathbb{R}^3 \tag{7.5}$$

and every $(R, \theta_0) \in \Gamma$, that is, if $F(\theta + \theta_0) = \rho_R F(\theta)$ for almost every θ, with the action ρ_R of (3.4). We write $\mathcal{H}^\Gamma_{\mathrm{div}}$ and $\mathcal{D}^\Gamma_S$ for the sets of the Γ-invariant elements of $\mathcal{H}_{\mathrm{div}}$ and of $\mathcal{D}_S$, and we let

$$m = 2 \text{ if } (R, \tfrac{1}{2}) \in \Gamma \text{ for some } R \in O(3), \qquad m = 1 \text{ otherwise.} \tag{7.6}$$

The number m enters the conditions (7.26) and (7.27): when $m = 2$, the norm $\|v(\theta)\|_{\dot{H}^1}$ of a field $v \in \mathcal{D}^\Gamma_S$ has period $\frac{1}{2}$. This follows from (7.5) because ρ_R preserves the $\dot{H}^1$ norm. For $R \in O(3)$ and $\theta_0 \in \mathbb{R}$ let $(\rho_{R,\theta_0} F)(\theta, y) = R\, F(\theta - \theta_0, R^{-1}y)$, so that F is Γ-invariant if and only if $\rho_{R,\theta_0} F = F$ for every $(R, \theta_0) \in \Gamma$. The map ρ_{R,θ_0} is an isometry of $\mathcal{H}$ onto itself; it commutes with $\mathbb{P}$ and with $\mathcal{T}_S$, preserves the relation $\simeq$, and satisfies $\rho_{R,\theta_0}[(F \cdot \nabla)G] = (\rho_{R,\theta_0} F \cdot \nabla)\rho_{R,\theta_0} G$. Hence the Γ-invariant fields form a closed subspace of $\mathcal{H}$, and $\mathcal{H}^\Gamma_{\mathrm{div}}$ is a closed subspace of $\mathcal{H}_{\mathrm{div}}$; the projection $\mathbb{P}$ maps Γ-invariant fields in $\mathcal{H}$ to Γ-invariant fields, and $(F \cdot \nabla)G$ is Γ-invariant if F and G are.

7.2. The approximate profiles and the problem for a perturbation. Let $n \geq 0$ be an integer, and let $\mathcal{K}$ be a compact convex subset of $\mathbb{R}^n$, the set of the parameters; its shape matters only in the last step of the proof, and it is specified in (7.29) below. For $\alpha \in \mathcal{K}$ we are given a period $S(\alpha)$ and a field $W^{\mathrm{a}}_\alpha : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, the *approximate profile*. The function S is continuous on $\mathcal{K}$, and $0 < S(\alpha) \leq S_{\max}$ for a number $S_{\max}$. The derivatives $\partial_\theta W^{\mathrm{a}}_\alpha$ and $\nabla W^{\mathrm{a}}_\alpha$ exist, and $W^{\mathrm{a}}_\alpha(\theta, y)$, $\partial_\theta W^{\mathrm{a}}_\alpha(\theta, y)$ and $\nabla W^{\mathrm{a}}_\alpha(\theta, y)$ are continuous functions of $(\alpha, \theta, y) \in \mathcal{K} \times \mathbb{R} \times \mathbb{R}^3$. For every $\alpha \in \mathcal{K}$ the field W^{a}_α has period 1 in θ and is divergence-free, its second derivatives in y, in the sense of distributions on $\mathbb{R} \times \mathbb{R}^3$, are locally integrable functions, and

$$W^{\mathrm{a}}_\alpha(\theta + \theta_0, y) = R\, W^{\mathrm{a}}_\alpha(\theta, R^{-1}y) \qquad \text{for all } (\theta, y) \in \mathbb{R} \times \mathbb{R}^3 \text{ and } (R, \theta_0) \in \Gamma. \tag{7.7}$$

Moreover W^{a}_α satisfies the bounds (1.3) of a profile, with a constant C_{a} in place of C_W which does not depend on α: for all $(\theta, y) \in \mathbb{R} \times \mathbb{R}^3$,

$$|W^{\mathrm{a}}_\alpha(\theta, y)| \leq C_{\mathrm{a}}\, (1 + |y|)^{-1}, \qquad |\nabla W^{\mathrm{a}}_\alpha(\theta, y)| \leq C_{\mathrm{a}}\, (1 + |y|)^{-2}. \tag{7.8}$$

The force that we add to the equation (7.1) is a combination of Γ-invariant fields $b_1, \dots, b_n \in \mathcal{H}$, the *forcing fields*, and $b_* \geq 0$ is a number such that

$$\sum_{i=1}^{n} \|b_i\|^2_{\mathcal{H}} \leq b_*^2. \tag{7.9}$$

We are given a continuous map $\beta^{\mathrm{a}} : \mathcal{K} \to \mathbb{R}^n$ and, for every $\alpha \in \mathcal{K}$, a field $F^{\mathrm{a}}_\alpha \in \mathcal{H}$, the *residual* of W^{a}_α, such that

$$\mathcal{T}_{S(\alpha)} W^{\mathrm{a}}_\alpha + (W^{\mathrm{a}}_\alpha \cdot \nabla) W^{\mathrm{a}}_\alpha + \sum_{i=1}^{n} \beta^{\mathrm{a}}_i(\alpha)\, b_i \simeq F^{\mathrm{a}}_\alpha. \tag{7.10}$$

The approximate equation includes the force $-\sum_i \beta^{\mathrm{a}}_i(\alpha) b_i$. The left side of (7.10) is locally integrable and may fail to be square integrable. The chosen representative F^{a}_α lies in $\mathcal{H}$, and (7.23) bounds its norm over a full period and all of $\mathbb{R}^3$, together with its pairings against specified fields. These bounds hold for every $\alpha \in \mathcal{K}$. Only the projected residual is determined

by (7.10). The representatives themselves may vary discontinuously with α and need not be Γ-invariant. Equation (7.58) gives continuity of their pairings with divergence-free test fields. The reference profiles and adjoint tests can be chosen separately at each parameter because only their estimates enter the fixed-point argument.

Time translation already gives a freedom in locating a periodic orbit. A phase condition fixes its origin in time. More generally, to allow for noninvertibility of the linearization we adjoin n unknown forcing coefficients and n linear constraints on the perturbation. Let $\Xi_1, \dots, \Xi_n \in \mathcal{H}$ be fields with values in the 3×3 matrices, and for a field v of period 1 in θ with $\nabla v \in \mathcal{H}$, where $(\nabla v)_{jk} = \partial_j v_k$, let

$$\ell_i(v) = \langle \nabla v, \Xi_i \rangle_{\mathcal{H}} \quad (1 \le i \le n), \qquad \ell(v) = (\ell_1(v), \dots, \ell_n(v)) \in \mathbb{R}^n. \tag{7.11}$$

Let $\bar{\ell} : \mathcal{K} \to \mathbb{R}^n$ be continuous. For $\alpha \in \mathcal{K}$ we look for a field $v \in \mathcal{D}^{\Gamma}_{S(\alpha)}$ and a vector $\beta \in \mathbb{R}^n$ such that

$$\mathcal{T}_{S(\alpha)} v + \mathcal{C}_\alpha v + (v \cdot \nabla) v + \sum_{i=1}^{n} \beta_i \, b_i \simeq -F^{\mathrm{a}}_\alpha, \qquad \ell(v) = \bar{\ell}(\alpha), \tag{7.12}$$

where

$$\mathcal{C}_\alpha v = (W^{\mathrm{a}}_\alpha \cdot \nabla) v + (v \cdot \nabla) W^{\mathrm{a}}_\alpha \tag{7.13}$$

collects the terms of the linearized equation that come from the approximate profile; the fields $\mathcal{C}_\alpha v$ and $(v \cdot \nabla) v$ belong to $\mathcal{H}$ (Lemmas 7.7 and 7.6). By (7.10), the first relation of (7.12) says that $\mathcal{W} = W^{\mathrm{a}}_\alpha + v$ solves the equation with the force $-\sum_i (\beta^{\mathrm{a}}_i(\alpha) + \beta_i)\, b_i$, that is,

$$\mathcal{T}_{S(\alpha)} \mathcal{W} + (\mathcal{W} \cdot \nabla) \mathcal{W} + \sum_{i=1}^{n} (\beta^{\mathrm{a}}_i(\alpha) + \beta_i)\, b_i \simeq 0,$$

so that $\mathcal{W}$ solves (7.1) modulo gradients if $\beta^{\mathrm{a}}(\alpha) + \beta = 0$.

7.3. The conditions of a certificate. Fix $\alpha \in \mathcal{K}$. The same constants work throughout $\mathcal{K}$; the residual estimates $\delta(\alpha)$ and $\delta_0(\alpha)$ may depend on α. The conditions serve four purposes.

Linear response, (7.14)–(7.21). The reference field W^{r}_α used in (7.18) may depend on α. The finite-data construction in Appendix B uses one field W^{r} throughout. Inequality (7.18) bounds the linearization at W^{r}_α, with separate allowances along the forcing components tested by e_k. The difference bound (7.19) and $\vartheta < 1$ transfer this estimate to W^{a}_α, giving injectivity with zero constraints. Surjectivity uses the identity-plus-compact representation in Proposition 7.8.

Size of the correction, (7.22)–(7.27). After transfer to W^{a}_α, (7.24) reserves ε for the residual and prescribed constraints. Under (7.25) the trial quadratic term fits in the remaining ε, giving $\|(v, \beta)\|_{w,\omega} \le 2\varepsilon$. The energy inequality (7.22) and (7.27) bound $\|\nabla v(\theta)\|_2$ by M at every phase. Together these bounds control $(v \cdot \nabla) v$ in $\mathcal{H}$. The joint argument retains (7.26) only for compatibility with the fixed-parameter proof.

Nonvanishing, (7.28). The approximate field has more gradient energy over a period than a correction in this ball can cancel.

Force cancellation, (7.29)–(7.37). Each β_i multiplies a forcing field b_i; Ξ_i specifies a constraint on v. The adjoint fields $\psi_{\alpha,j}$ and matrix $Y(\alpha)$ relate force components to those constraints. The columns of V test the total forcing vector in (7.30). For a trial quadratic term the update is projected onto $\mathcal{K}$. Once the trial term equals $(v \cdot \nabla) v$, (7.35) and (7.37) put the unprojected update in $\mathcal{K}$. At a fixed point it therefore equals α, and invertibility of V makes every force coefficient zero.

Perturbations are measured with a weight. Let $w : \mathbb{R}^3 \to \mathbb{R}$ be a bounded measurable function with $w \geq w_*$ for a number $w_* > 0$, and let $\omega > 0$. For a field $u \in X$ with $\Delta u \in L^2$, for $v \in \mathcal{D}_S$ and for $\beta \in \mathbb{R}^n$ we write

$$|u|_w^2 = \int_{\mathbb{R}^3} (|\Delta u|^2 + w\,|\nabla u|^2)\,dy, \qquad \|v\|_w^2 = \int_0^1 |v(\theta)|_w^2\,d\theta, \tag{7.14}$$
$$\|(v,\beta)\|_{w,\omega}^2 = \|v\|_w^2 + \omega^2|\beta|^2,$$

where $|\beta|$ is the Euclidean norm, and we let

$$c_w = (4\pi\sqrt{w_*})^{-1/2}, \tag{7.15}$$

which is the constant of the inequality $\|u\|_{L^\infty} \leq c_w\,|u|_w$ (Lemma 7.6).

The first condition is an a priori estimate for the linearization of (7.12). It is assumed for a second family of fields, the *reference profiles*, which may be simpler than the approximate profiles and are close to them by (7.19) below. For every $\alpha \in \mathcal{K}$ let $W_\alpha^{\mathrm{r}} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1, with period 1 in θ, and for $v \in \mathcal{D}_{S(\alpha)}$ let

$$\mathcal{C}_\alpha^{\mathrm{r}} v = (W_\alpha^{\mathrm{r}} \cdot \nabla)v + (v \cdot \nabla)W_\alpha^{\mathrm{r}}, \tag{7.16}$$

a locally integrable field. The right-hand side of the linearized equation is measured in $\mathcal{H}$, and separately in finitely many directions, so that the constant of the estimate may be larger on finitely many components of the right-hand side than on the others. Let $N \geq 0$ be an integer, let $c_0 \geq 0$, and let $e_1, \dots, e_N \in \mathcal{H}$ be pairwise orthogonal fields with $\|e_k\|_{\mathcal{H}} \leq c_0$; for $F \in \mathcal{H}$ we write

$$[F] = \left(\sum_{k=1}^{N} \langle F, e_k \rangle_{\mathcal{H}}^2 \right)^{1/2}, \qquad \text{so that} \qquad [F] \leq c_0\,\|F\|_{\mathcal{H}} \tag{7.17}$$

by Bessel's inequality. Let $c_1 \geq 0$ and $c_2 \in \mathbb{R}$. We assume that

$$\|(v,\beta)\|_{w,\omega} \leq c_1\,\|F\|_{\mathcal{H}} + [F] + c_2\,|\ell(v)| \tag{7.18}$$

for every $v \in \mathcal{D}^\Gamma_{S(\alpha)}$, every $\beta \in \mathbb{R}^n$ and every $F \in \mathcal{H}$ with $F \simeq \mathcal{T}_{S(\alpha)} v + \mathcal{C}_\alpha^{\mathrm{r}} v + \sum_i \beta_i\, b_i$. Here, and in (7.33) below, the field F need not be divergence-free, and the norm and the seminorm are those of F and not of $\mathbb{P}F$. Let $c_{\mathrm{r}} \geq 0$ and $c_L > 0$. We assume that the reference profile is close to the approximate profile: for every $v \in \mathcal{D}^\Gamma_{S(\alpha)}$, the field $\mathcal{C}_\alpha v - \mathcal{C}_\alpha^{\mathrm{r}} v$ belongs to $\mathcal{H}$ and

$$\|((W_\alpha^{\mathrm{a}} - W_\alpha^{\mathrm{r}}) \cdot \nabla)v + (v \cdot \nabla)(W_\alpha^{\mathrm{a}} - W_\alpha^{\mathrm{r}})\|_{\mathcal{H}} \leq c_{\mathrm{r}}\,\|v\|_w. \tag{7.19}$$

With

$$\vartheta = (c_1 + c_0)\,c_{\mathrm{r}} \tag{7.20}$$

we assume, moreover, that

$$\vartheta < 1, \qquad c_1 + c_0 \leq (1 - \vartheta)\,c_L. \tag{7.21}$$

Apply (7.18) to $F - (\mathcal{C}_\alpha - \mathcal{C}_\alpha^{\mathrm{r}})v$. By (7.17) and (7.19), the additional term is at most $\vartheta\|v\|_w \leq \vartheta\|(v,\beta)\|_{w,\omega}$. Moving it to the left gives the estimate with factor $1 - \vartheta$; (7.21) then yields $\|(v,\beta)\|_{w,\omega} \leq c_L\|F\|_{\mathcal{H}}$ when $\ell(v) = 0$. In the case $W_\alpha^{\mathrm{r}} = W_\alpha^{\mathrm{a}}$, $c_{\mathrm{r}} = 0$, $N = 0$ and $\omega = 1$, one has $\vartheta = 0$ and $[F] = 0$. Then (7.18) directly estimates the equation linearized about W_α^{a}.

The norm $\|\cdot\|_w$ controls the mean of $\|v(\theta)\|_{\dot{H}^1}^2$ over a period, and the bound for the quadratic term of (7.12) (Lemma 7.6) uses its supremum. For this we assume a lower bound for a quadratic form at each time, with a number $c_3 \geq 0$:

$$\int_{\mathbb{R}^3} (|\Delta u|^2 + \tfrac{1}{4}\,|\nabla u|^2)\,dy - \int_{\mathbb{R}^3} [(W_\alpha^{\mathrm{a}}(\theta) \cdot \nabla)u + (u \cdot \nabla)W_\alpha^{\mathrm{a}}(\theta)] \cdot \Delta u\,dy \geq -c_3\,|u|_w^2 \tag{7.22}$$

for every $\theta \in \mathbb{R}$ and every $u \in X$ with $\Delta u \in L^2$ and $\rho_R u = u$ for all $(R,0) \in \Gamma$. Only symmetries with zero time shift enter this condition on a single spatial field. For a solution v of the linearized equation, (7.42) expresses the derivative of $\|v(\theta)\|_{\dot H^1}^2/(2S(\alpha))$ as minus the left side of (7.22), together with pairings of the right-hand side and forcing fields with $\Delta v(\theta)$. Integrating from a minimum of the norm bounds its maximum over a period, or over half a period when $m = 2$ (Proposition 7.9).

The solution of (7.12) is found among the pairs (v,β) with $\|(v,\beta)\|_{w,\omega} \le 2\varepsilon$ and $\sup_\theta \|v(\theta)\|_{\dot H^1} \le M$, for two numbers $\varepsilon \ge 0$ and $M > 0$. Let $\delta, \delta_0 : \mathcal{K} \to \mathbb{R}$ and $\delta_{\max} \in \mathbb{R}$. The residual has two bounds in (7.23). A residual orthogonal to every e_k has zero contribution to $[F_\alpha^{\mathrm{a}}]$ even when its norm is positive. Keeping these pairings separate can leave room in (7.24) that would be lost on replacing them by $c_0\|F_\alpha^{\mathrm{a}}\|_{\mathcal{H}}$. The residual and constraints use at most ε after the linear solve, leaving another ε for the quadratic term. We require

$$\|F_\alpha^{\mathrm{a}}\|_{\mathcal{H}} \le \delta(\alpha) \le \delta_{\max}, \qquad [F_\alpha^{\mathrm{a}}] \le \delta_0(\alpha), \tag{7.23}$$

$$c_1\,\delta(\alpha) + \delta_0(\alpha) + c_2\,|\bar\ell(\alpha)| \le (1-\vartheta)\,\varepsilon. \tag{7.24}$$

For a pair of this set the quadratic term has norm at most $2\,c_w\,\varepsilon\,M$ in $\mathcal{H}$ (Lemma 7.6), and the solution of the linearized problem with a Γ-invariant field of this size in place of the quadratic term will again belong to the set. We assume that, with the numbers m of (7.6) and c_w of (7.15),

$$2\,c_L\,c_w\,M < 1, \tag{7.25}$$

$$4\,\varepsilon^2\left[\frac{1}{w_*} + \frac{2S_{\max}}{m}\left(c_3 + \frac{b_*}{\omega} + \frac{1}{c_L}\right)\right] \le M^2, \tag{7.26}$$

$$4\,\varepsilon^2\left[\frac{1}{w_*} + \frac{2S_{\max}}{m}c_3\right] + \frac{4S_{\max}}{m}\,\varepsilon\left[\delta_{\max} + 2\,c_w\,\varepsilon\,M + \frac{2\,b_*}{\omega}\,\varepsilon\right] \le M^2. \tag{7.27}$$

Conditions (7.24) and (7.25) give the bound 2ε in the norm of pairs. Condition (7.27) gives the uniform-in-time bound M. The fixed-parameter contraction in the formal proof uses (7.26) to control the time supremum of the difference between two corrections. The joint fixed-point argument below uses (7.27). To exclude the zero solution we also require

$$4\,\varepsilon^2 < w_*\,\|\nabla W_\alpha^{\mathrm{a}}\|_{\mathcal{H}}^2. \tag{7.28}$$

Indeed, (7.14) gives $w_*\|\nabla v\|_{\mathcal{H}}^2 \le 4\varepsilon^2$, so (7.28) excludes $v = -W_\alpha^{\mathrm{a}}$. The remaining conditions are used to find a parameter at which the force vanishes. Let $\mathcal{J}$ be a subset of $\{1,\dots,n\}$, and let $\varrho_j \ge 0$ for $j \in \mathcal{J}$, $\varrho_* \ge 0$, and $d_j \in \mathbb{R}\setminus\{0\}$ for $1 \le j \le n$. The parameter set is

$$\mathcal{K} = \left\{\alpha \in \mathbb{R}^n : |\alpha_j| \le \varrho_j \text{ for } j \in \mathcal{J}, \quad \sum_{j\notin\mathcal{J}} d_j^2\alpha_j^2 \le \varrho_*^2\right\}, \tag{7.29}$$

a compact convex set that contains the origin. Let V be an invertible real $n\times n$ matrix, with the columns $V_1,\dots,V_n \in \mathbb{R}^n$. In the last step of the proof we associate with $\alpha \in \mathcal{K}$ and $\beta \in \mathbb{R}^n$ the vector $\alpha' \in \mathbb{R}^n$ with the coordinates

$$\alpha_j' = \alpha_j - d_j^{-1}\,V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) \qquad (1 \le j \le n). \tag{7.30}$$

If $\alpha' = \alpha$, then $V_j \cdot (\beta^{\mathrm{a}}(\alpha)+\beta) = 0$ for every j, and $\beta^{\mathrm{a}}(\alpha)+\beta = 0$, because V is invertible. The change in the jth coordinate is minus the scalar product of the total forcing vector with the column $d_j^{-1}V_j$. In Proposition 7.10 the linear solve determines β from both α and a trial quadratic term. The update (7.30) enters that joint fixed-point map. The conditions below ensure that $\alpha' \in \mathcal{K}$ whenever (v,β) is a solution of (7.12) at α with the two bounds above,

that is, by (7.29), that the numbers $V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j\alpha_j = -d_j\alpha'_j$ are at most $|d_j|\,\varrho_j$ in absolute value for $j \in \mathcal{J}$ and that the sum of their squares over $j \notin \mathcal{J}$ is at most ϱ_*^2. For each coordinate in $\mathcal{J}$ we test (7.12) against a separate field. For the remaining coordinates we use the single Euclidean bound $\omega|\beta| \le 2\varepsilon$.

For $j \in \mathcal{J}$ and $\alpha \in \mathcal{K}$ let $\psi_{\alpha,j} \in \mathcal{H}$, and for every $\alpha \in \mathcal{K}$ let $Y(\alpha)$ be a real $n \times n$ matrix; nothing is assumed of the dependence of $\psi_{\alpha,j}$ and of $Y(\alpha)$ on α. We write

$$\langle b, \psi_{\alpha,j}\rangle_{\mathcal{H}} = \big(\langle b_1, \psi_{\alpha,j}\rangle_{\mathcal{H}}, \dots, \langle b_n, \psi_{\alpha,j}\rangle_{\mathcal{H}}\big) \in \mathbb{R}^n. \tag{7.31}$$

Let $a_j \ge 0$, $\tilde{a}_j \ge 0$ and $o_j \in \mathbb{R}$ for $j \in \mathcal{J}$. We assume that, for every $j \in \mathcal{J}$ and every $v \in \mathcal{D}^{\Gamma}_{S(\alpha)}$,

$$|\langle (v\cdot\nabla)v, \psi_{\alpha,j}\rangle_{\mathcal{H}}| \le \tilde{a}_j \,\|v\|_w^2 \tag{7.32}$$

and, for every $F \in \mathcal{H}$ with $F \simeq \mathcal{T}_{S(\alpha)}v + \mathcal{C}_\alpha v$,

$$\Big|\langle F, \psi_{\alpha,j}\rangle_{\mathcal{H}} - \big(Y(\alpha)\,\ell(v)\big)_j\Big| \le a_j\,\|v\|_w, \tag{7.33}$$

and that, for every $j \in \mathcal{J}$,

$$\Big|V_j \cdot \beta^{\mathrm{a}}(\alpha) - \langle F^{\mathrm{a}}_\alpha, \psi_{\alpha,j}\rangle_{\mathcal{H}} - \big(Y(\alpha)\,\bar{\ell}(\alpha)\big)_j - d_j\alpha_j\Big| \le o_j, \tag{7.34}$$

$$\tilde{a}_j\,(2\varepsilon)^2 + 2\,\varepsilon\left(a_j^2 + \frac{|\langle b, \psi_{\alpha,j}\rangle_{\mathcal{H}} - V_j|^2}{\omega^2}\right)^{1/2} + o_j \le |d_j|\,\varrho_j. \tag{7.35}$$

The condition (7.33) requires that $\psi_{\alpha,j}$ be an approximate solution of the adjoint of the linearized problem; it includes the case $v = 0$, in which it says that $\langle F, \psi_{\alpha,j}\rangle_{\mathcal{H}} = 0$ for every $F \in \mathcal{H}$ with $F \simeq 0$, that is, that $\psi_{\alpha,j} \in \mathcal{H}_{\mathrm{div}}$. For a solution (v, β) of (7.12), the pairing of (7.12) with $\psi_{\alpha,j}$ then gives

$$\beta \cdot \langle b, \psi_{\alpha,j}\rangle_{\mathcal{H}} = -\langle F^{\mathrm{a}}_\alpha, \psi_{\alpha,j}\rangle_{\mathcal{H}} - \big(Y(\alpha)\,\bar{\ell}(\alpha)\big)_j$$

up to an error of at most $a_j\,\|v\|_w + \tilde{a}_j\,\|v\|_w^2$, by (7.33) and (7.32). The left-hand side is $V_j \cdot \beta$ up to the term $\beta \cdot (\langle b, \psi_{\alpha,j}\rangle_{\mathcal{H}} - V_j)$, which is small when the column V_j is close to the vector (7.31). Hence (7.34) bounds $V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j\alpha_j$ up to these errors. Cauchy–Schwarz applied to $(\|v\|_w, \omega|\beta|)$ and $(a_j, |\langle b, \psi_{\alpha,j}\rangle_{\mathcal{H}} - V_j|/\omega)$ gives the square-root term in (7.35); the quadratic error is at most $\tilde{a}_j(2\varepsilon)^2$. For the coordinates outside $\mathcal{J}$ let $a_* \ge 0$ and $o_* \in \mathbb{R}$; we assume that

$$\sum_{j\notin\mathcal{J}} (V_j\cdot\beta)^2 \le a_*^2\,|\beta|^2 \quad \text{for every } \beta \in \mathbb{R}^n, \qquad \left(\sum_{j\notin\mathcal{J}} \big(V_j\cdot\beta^{\mathrm{a}}(\alpha) - d_j\alpha_j\big)^2\right)^{1/2} \le o_*, \tag{7.36}$$

$$\frac{2\,\varepsilon\,a_*}{\omega} + o_* \le \varrho_*. \tag{7.37}$$

A *certificate* consists of the set Γ and of the objects introduced in this subsection and in Subsection 7.2, with the properties stated there, such that the following hold for every $\alpha \in \mathcal{K}$: the a priori estimate (7.18) with (7.19) and (7.21); the lower bound (7.22); the bounds (7.23), (7.24) and (7.28) for the residual and the approximate profile; the inequalities (7.25) to (7.27) for ε and M; and the conditions (7.32) to (7.37) on the parameter. The integers n and N may be zero, and the set $\mathcal{K}$ may have empty interior. The definition does not prescribe the form in which the fields of a certificate are given; the form that the formalization prescribes for them is described in Appendix B.1.

7.4. The certificate theorem.

Theorem 7.1. *Suppose that a certificate is given. Then there exist $\alpha \in \mathcal{K}$ and a profile (W, q) with period $S = S(\alpha)$ and isometry I, in the sense of (1.2) and (1.3), such that the field $v(\theta, y) = W(S\theta, y) - W^{\mathrm{a}}_\alpha(\theta, y)$ belongs to $\mathcal{D}^\Gamma_S$ and satisfies*

$$\|v\|_w^2 + \omega^2 \, |\beta^{\mathrm{a}}(\alpha)|^2 \le 4\,\varepsilon^2, \qquad \sup_{\theta\in\mathbb{R}} \|v(\theta)\|_{\dot H^1} \le M, \qquad \ell(v) = \bar\ell(\alpha). \tag{7.38}$$

Moreover,

$$W(s + \theta_0 S, y) = R\, W(s, R^{-1} y) \qquad \text{for all } (s, y) \in \mathbb{R}\times\mathbb{R}^3 \text{ and every } (R, \theta_0) \in \Gamma. \tag{7.39}$$

Corollary 7.2. *If there exists a certificate, then the conclusions of Theorem 1.1 hold, on $\mathbb{R}^3$ and on $\mathbb{T}^3$, with the profile (W, q) of Theorem 7.1, the period $S(\alpha)$ and the isometry I.*

Remark 7.3. For $(R, \theta_0) \in \Gamma$ with $\theta_0 > 0$, the relation (7.39) is the last equation of (1.2) with the period $\theta_0 S$ and the isometry R, and neither the other equations of (1.2) nor the bounds (1.3) involve the period or the isometry. Hence the profile (W, q) of Theorem 7.1 is also a profile with period $\theta_0 S$ and isometry R; in particular, when $m = 2$, with period $S/2$ and isometry R for every R such that $(R, \frac12) \in \Gamma$.

The term $\omega^2|\beta^{\mathrm{a}}(\alpha)|^2$ in (7.38) measures the force coefficients removed by the correction, since $\beta = -\beta^{\mathrm{a}}(\alpha)$ at the selected parameter. For the bounds specified in Appendix B, $M < 0.017$. At the parameter selected by the theorem, (7.38) therefore bounds the L^2 norm of $\nabla W(S\theta) - \nabla W^{\mathrm{a}}_\alpha(\theta)$ by 0.017 at every phase. The comparison is with this member of the family. Figure 1 shows the field $\bar W$ used to form the family.

Fix a certificate for the proof of Theorem 7.1. The constants of the certificate are written out in every estimate that leads to one of its conditions. The other constants, denoted by C or implied by the symbol $\lesssim$, depend only on C_{a}, unless we say otherwise, and they enter none of the conditions.

7.5. The free equation. To study $\mathcal{T}_S$, we use the time-dependent dilation that removes its drift term. For $S > 0$ and a locally integrable field v on $\mathbb{R}\times\mathbb{R}^3$, set

$$t = 1 - e^{-S\theta}, \qquad x = e^{-S\theta/2}\, y, \qquad u(t, x) = e^{S\theta/2}\, v(\theta, y). \tag{7.40}$$

These are the Eulerian variables of (1.4) with $T = 1$, for the self-similar time $s = S\theta$ and with v in place of W; thus (t, x) ranges over $(-\infty, 1)\times\mathbb{R}^3$, and $u(t) = D_{1/(1-t)}\, v(\theta)$ with the scaling D_μ of (3.4). By the chain rule,

$$\partial_t u(t, x) - \Delta u(t, x) = e^{3S\theta/2}\, \mathcal{T}_S v(\theta, y), \qquad \Delta u(t, x) = e^{3S\theta/2}\, \Delta v(\theta, y) \tag{7.41}$$

in the sense of distributions on $(-\infty, 1)\times\mathbb{R}^3$. Thus $\mathcal{T}_S$ becomes the heat operator, and the identity (7.42) becomes the energy identity for the heat equation.

Lemma 7.4. *Let $S > 0$ and $v \in \mathcal{D}_S$. Then $\mathcal{T}_S v \in \mathcal{H}_{\mathrm{div}}$. After a modification on a set of times of measure zero, v is a continuous map from $\mathbb{R}$ to X of period 1, the function $\theta \mapsto \|v(\theta)\|^2_{\dot H^1}$ is locally absolutely continuous, and for almost every θ*

$$\frac{1}{2S}\frac{d}{d\theta}\|v(\theta)\|^2_{\dot H^1} + \|\Delta v(\theta)\|^2_{L^2} + \tfrac14\,\|v(\theta)\|^2_{\dot H^1} = -\int_{\mathbb{R}^3} \mathcal{T}_S v(\theta)\cdot\Delta v(\theta)\, dy. \tag{7.42}$$

In particular,

$$\|\Delta v\|^2_{\mathcal{H}} + \tfrac14\,\|\nabla v\|^2_{\mathcal{H}} = -\langle \mathcal{T}_S v, \Delta v\rangle_{\mathcal{H}} \le \|\mathcal{T}_S v\|^2_{\mathcal{H}}. \tag{7.43}$$

Proof. The field $\mathcal{T}_S v$ belongs to $\mathcal{H}$, by (7.4), and it is divergence-free, because $\operatorname{div} v = 0$ and $\operatorname{div}[(y\cdot\nabla)v] = (y\cdot\nabla)\operatorname{div} v + \operatorname{div} v$.

Let u be the field (7.40). Since $dx = e^{-3S\theta/2}\,dy$, it follows from (7.40) and (7.41) that for almost every θ

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 &= e^{S\theta/2}\,\|\nabla v(\theta)\|_{L^2}^2, \\ \int_{\mathbb{R}^3} \partial_t u(t)\cdot\Delta u(t)\,dx &= e^{3S\theta/2}\int_{\mathbb{R}^3}(\mathcal{T}_S v(\theta)+\Delta v(\theta))\cdot\Delta v(\theta)\,dy. \end{aligned} \tag{7.44}$$

On each compact time interval below 1, the factors in (7.44) and $dt/d\theta = Se^{-S\theta}$ are bounded above and below by positive numbers. Thus (7.4) gives $u \in L^2_{\mathrm{loc}}((-\infty,1);X)$ and $\partial_t u, \Delta u \in L^2_{\mathrm{loc}}((-\infty,1);L^2(\mathbb{R}^3;\mathbb{R}^3))$.

We claim that, after a modification on a set of times of measure zero, u is a continuous map from $(-\infty,1)$ to X, and

$$\|\nabla u(b)\|_{L^2}^2 - \|\nabla u(a)\|_{L^2}^2 = -2\int_a^b\int_{\mathbb{R}^3}\partial_t u\cdot\Delta u\,dx\,dt \qquad (a<b<1). \tag{7.45}$$

Use the Hilbert triple $H^1 \subset L^2 \subset H^{-1}$ of matrix fields on $\mathbb{R}^3$. The Lions–Magenes lemma gives continuity into L^2 and the integration-by-parts identity for the squared norm [72, p. 180, (1.12)–(1.13)], [85, Ch. III, Lemma 1.2]. We apply it with ∇u in both slots [99, Appendix 6.B, Theorems 6.41–6.42, pp. 208–209]. By (3.2) and Plancherel, $\|\nabla^2 u(t)\|_{L^2} = \|\Delta u(t)\|_{L^2}$, so $\nabla u \in L^2((a,b);H^1)$. The distribution $\partial_t \nabla u$ acts on a matrix test field Ψ as $-\int \partial_t u\cdot\operatorname{div}\Psi$; hence it belongs to $L^2((a,b);H^{-1})$. Its pairing with ∇u is $-\int \partial_t u\cdot\Delta u$. The lemma gives continuity in L^2 and (7.45). Since gradients of fields in X form a closed subspace of L^2, this representative determines a continuous X-valued representative of u.

We return to v. By (7.40), $v(\theta) = D_{e^{-S\theta}}\,u(1-e^{-S\theta})$, and the map $(\mu,u)\mapsto D_\mu u$ is continuous from $(0,\infty)\times X$ to X (Subsection 3.1). This gives a continuous X-valued representative of v. Its period is 1: the continuous difference $v(\theta+1)-v(\theta)$ vanishes almost everywhere and therefore everywhere. By (7.44) and (7.45), the function $\|v(\theta)\|_{\dot H^1}^2 = e^{-S\theta/2}\,\|\nabla u(1-e^{-S\theta})\|_{L^2}^2$ is locally absolutely continuous, and for almost every θ

$$\frac{d}{d\theta}\|v(\theta)\|_{\dot H^1}^2 = -\frac{S}{2}\|v(\theta)\|_{\dot H^1}^2 - 2S\int_{\mathbb{R}^3}(\mathcal{T}_S v(\theta)+\Delta v(\theta))\cdot\Delta v(\theta)\,dy.$$

This is (7.42); the term $\frac14\|v(\theta)\|_{\dot H^1}^2$ comes from the scaling. Finally we integrate (7.42) over a period. The integral of the first term is zero, and we obtain the equality in (7.43). By the Cauchy–Schwarz inequality it implies $\|\Delta v\|_{\mathcal{H}} \le \|\mathcal{T}_S v\|_{\mathcal{H}}$, and the inequality in (7.43) follows. □

When S varies, the coefficient of $\partial_\theta v$ in (7.4) varies with it. Passing from $v\in\mathcal{D}_S$ to $f = \mathcal{T}_S v$ puts solutions for different periods in the same Hilbert space $\mathcal{H}_{\mathrm{div}}$.

We construct the periodic inverse by integrating the forcing against the free flow $D_{e^{-s}}e^{(1-e^{-s})\Delta}$. Its norm on X is at most $e^{-s/4}$, by (3.3) and (3.5), which makes the integral over past times converge. Estimate (7.43) then extends the construction from the dense test fields to all of $\mathcal{H}_{\mathrm{div}}$.

Proposition 7.5. *Let $S>0$. The operator $\mathcal{T}_S$ is a bijection of $\mathcal{D}_S$ onto $\mathcal{H}_{\mathrm{div}}$, and of $\mathcal{D}_S^\Gamma$ onto $\mathcal{H}_{\mathrm{div}}^\Gamma$. Moreover, if $S_k\to S$ in $(0,\infty)$ and $f_k\to f$ weakly in $\mathcal{H}_{\mathrm{div}}$, then $v_k = \mathcal{T}_{S_k}^{-1}f_k$ and $v = \mathcal{T}_S^{-1}f$ satisfy $\nabla v_k\to\nabla v$ and $\Delta v_k\to\Delta v$ weakly in $\mathcal{H}$, and $v_k\to v$ in $L^2((0,1)\times B_r)$ for every $r>0$.*

Proof. If $v \in \mathcal{D}_S$ and $\mathcal{T}_S v = 0$, then $\nabla v = 0$ by (7.43), and $v = 0$ by the Sobolev inequality (3.1). Thus $\mathcal{T}_S$ is injective.

We next show that the range of $\mathcal{T}_S$ contains the products (7.3). Let $\psi \in C^\infty_{c,\mathrm{div}}$, and for $s \ge 0$ let

$$U(s) = D_{e^{-s}}\, e^{(1-e^{-s})\Delta}\, \psi, \qquad \text{that is,} \qquad U(s,y) = e^{-s/2}\, \big(e^{(1-e^{-s})\Delta}\, \psi\big)(e^{-s/2}\, y).$$

This is a smooth divergence-free field on $[0,\infty) \times \mathbb{R}^3$ with $U(0) = \psi$. In the variables (7.40), with $S = 1$ and with (s, U) in place of (θ, v), it is the solution $e^{t\Delta}\psi$ of the heat equation on $[0,1) \times \mathbb{R}^3$, so that the chain rule (7.41) gives

$$\partial_s U + \tfrac{1}{2}\, U + \tfrac{1}{2}\, (y \cdot \nabla) U - \Delta U = 0. \tag{7.46}$$

The heat flow does not increase the norms of the gradient and of the Laplacian in L^2, and D_μ multiplies them by $\mu^{1/4}$ and by $\mu^{3/4}$. Hence

$$\|\nabla U(s)\|_{L^2} \le e^{-s/4}\, \|\nabla \psi\|_{L^2}, \qquad \|\Delta U(s)\|_{L^2} \le e^{-3s/4}\, \|\Delta \psi\|_{L^2}, \tag{7.47}$$

and U and its derivatives in y are bounded on $[0,\infty) \times \mathbb{R}^3$ by multiples of $e^{-s/2}$, because the derivatives of $e^{t\Delta}\psi$ are bounded on $[0,1] \times \mathbb{R}^3$. Let $a \in C^\infty(\mathbb{R})$ have period 1, and let

$$v(\theta) = \int_0^\infty a(\theta - \tfrac{s}{S})\, U(s)\, ds.$$

Then v is a smooth divergence-free field on $\mathbb{R} \times \mathbb{R}^3$ of period 1 in θ, and by (7.47) it is a continuous map from $\mathbb{R}$ to X with Δv continuous from $\mathbb{R}$ to L^2. Since a and its derivatives are bounded and U decays exponentially, we may differentiate under the integral and integrate by parts in s. The endpoint at infinity vanishes; $U(0) = \psi$ gives the term $a(\theta)\psi$. Using (7.46),

$$\begin{aligned} \tfrac{1}{S}\partial_\theta v(\theta) &= -\int_0^\infty \frac{d}{ds}\big[a(\theta - \tfrac{s}{S})\big]\, U(s)\, ds = a(\theta)\, \psi + \int_0^\infty a(\theta - \tfrac{s}{S})\, \partial_s U(s)\, ds \\ &= a(\theta)\, \psi - \tfrac{1}{2}\, v(\theta) - \tfrac{1}{2}\, (y \cdot \nabla) v(\theta) + \Delta v(\theta). \end{aligned}$$

We have $\mathcal{T}_S v = a(\theta)\psi(y)$ and $v \in \mathcal{D}_S$. Linearity puts the dense span of (7.3) in the range. For f in $\mathcal{H}_{\mathrm{div}}$ choose $v_k \in \mathcal{D}_S$ with $\mathcal{T}_S v_k \to f$. By (7.43) on $v_k - v_l$, ∇v_k and Δv_k are Cauchy in $\mathcal{H}$. Their limits give a period-1 field $v \in L^2_{\mathrm{loc}}(\mathbb{R}; X)$ with $\Delta v \in \mathcal{H}$. In distributions $\mathcal{T}_S v = f$, so $v \in \mathcal{D}_S$, proving surjectivity.

Let $(R, \theta_0) \in \Gamma$. The map ρ_{R,θ_0} of Subsection 7.1 maps $\mathcal{D}_S$ onto itself and commutes with $\mathcal{T}_S$. Hence $\mathcal{T}_S$ maps $\mathcal{D}_S^\Gamma$ into $\mathcal{H}_{\mathrm{div}}^\Gamma$. Conversely, if $f \in \mathcal{H}_{\mathrm{div}}^\Gamma$ and $v = \mathcal{T}_S^{-1} f$, then $\mathcal{T}_S \rho_{R,\theta_0} v = \rho_{R,\theta_0} f = f$, so that $\rho_{R,\theta_0} v = v$ by the injectivity; thus $v \in \mathcal{D}_S^\Gamma$.

For the last assertion, weak convergence bounds f_k, and $S_k \to S > 0$ keeps the periods in a compact subset of $(0,\infty)$. Equations (7.43) and (3.1) bound $\nabla v_k, \Delta v_k$ in $\mathcal{H}$ and v_k in $L^2((0,1); L^6)$. On $(0,1) \times B_r$, the latter bound controls v_k in L^2, and $|(y \cdot \nabla) v_k| \le r |\nabla v_k|$. Hence

$$\partial_\theta v_k = S_k (f_k + \Delta v_k - \tfrac{1}{2}\, v_k - \tfrac{1}{2}\, (y \cdot \nabla) v_k)$$

is bounded in $L^2((0,1) \times B_r)$. The spatial and time derivative bounds give local H^1 bounds in four-dimensional space-time. Periodicity extends them across the time endpoints. Rellich's theorem on bounded balls [14, Theorem 9.16, p. 285], with $p = q = 2$ and dimension 4, gives precompactness in $L^2((0,1) \times B_r)$ for every $r > 0$. Extract weak limits $v_k \rightharpoonup \tilde{v}$ in $L^2((0,1); X)$ and $\Delta v_k \rightharpoonup \Delta \tilde{v}$ in $\mathcal{H}$. In distributions $\mathcal{T}_{S_k} v_k = f_k$ gives $\mathcal{T}_S \tilde{v} = f$, so $\tilde{v} \in \mathcal{D}_S$ equals v by injectivity. This identifies every subsequential limit. Hence ∇v_k and Δv_k converge weakly for the full sequence, and local precompactness gives $v_k \to v$ in $L^2((0,1) \times B_r)$ for every $r > 0$. □

7.6. The quadratic term and the terms from the approximate profile.

Lemma 7.6. *Let $u \in X$ with $\Delta u \in L^2$. Then*

$$\|u\|_{L^\infty}^2 \le c_w^2 \int_{\mathbb{R}^3} (|\Delta u|^2 + w_* |\nabla u|^2)\, dy. \tag{7.48}$$

Consequently, for $S > 0$ and $v \in \mathcal{D}_S$ the field $(v \cdot \nabla)v$ belongs to $\mathcal{H}$ and, with the norm $\|v\|_w$ of (7.14),

$$\|(v \cdot \nabla)v\|_{\mathcal{H}} \le c_w \|v\|_w \sup_{\theta \in \mathbb{R}} \|v(\theta)\|_{\dot{H}^1}. \tag{7.49}$$

Proof. The identification in Subsection 3.1 puts u in the homogeneous Dirichlet completion of $C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$. Together with $\Delta u \in L^2$, this is precisely the hypothesis of Xie's inequality [96, Theorem, p. 1185]; its vector-valued form is stated in [97, Theorem 1 and Section 2, preprint p. 2]. Square-integrability of u itself is unnecessary. The arithmetic–geometric mean inequality then gives

$$\|u\|_{L^\infty}^2 \le \frac{1}{2\pi} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \le \frac{1}{4\pi\sqrt{w_*}} (\|\Delta u\|_{L^2}^2 + w_* \|\nabla u\|_{L^2}^2),$$

which is (7.48), by (7.15). Let now $S > 0$ and $v \in \mathcal{D}_S$. By Lemma 7.4, v is continuous and periodic with values in X, so that the supremum in (7.49) is finite. For almost every θ the field $v(\theta)$ satisfies the hypotheses of the first part, and $w \ge w_*$; hence $\|v(\theta)\|_{L^\infty} \le c_w |v(\theta)|_w$ by (7.48). Since $|(v \cdot \nabla)v| \le |v|\,|\nabla v|$,

$$\|(v \cdot \nabla)v\|_{\mathcal{H}}^2 \le \int_0^1 \|v(\theta)\|_{L^\infty}^2 \|v(\theta)\|_{\dot{H}^1}^2 \, d\theta \le c_w^2 \|v\|_w^2 \sup_{\theta \in \mathbb{R}} \|v(\theta)\|_{\dot{H}^1}^2.$$

□

In (7.49) the period norm controls $\|v(\theta)\|_\infty$ in L^2 of time, while M controls $\|\nabla v(\theta)\|_2$ in L^∞ of time. Their product bounds the space-time L^2 norm of the quadratic forcing.

Lemma 7.7. *Let $\alpha \in \mathcal{K}$ and $S = S(\alpha)$. There exists a constant C, which depends only on C_{a}, such that for every $v \in L^2_{\mathrm{loc}}(\mathbb{R}; X)$ of period 1 in θ the field $\mathcal{C}_\alpha v$ of (7.13) belongs to $\mathcal{H}$ and $\|\mathcal{C}_\alpha v\|_{\mathcal{H}} \le C \|\nabla v\|_{\mathcal{H}}$. If v is Γ-invariant, so is $\mathcal{C}_\alpha v$. The operator $\mathcal{C}_\alpha \mathcal{T}_S^{-1}$ from $\mathcal{H}_{\mathrm{div}}$ to $\mathcal{H}$ is compact.*

Proof. By (7.8),

$$|\mathcal{C}_\alpha v| \le C_{\mathrm{a}} \left((1 + |y|)^{-1} |\nabla v| + (1 + |y|)^{-2} |v|\right). \tag{7.50}$$

The function $(1 + |y|)^{-2}$ belongs to $L^3(\mathbb{R}^3)$. By Hölder's inequality and the Sobolev inequality (3.1), for every $r \ge 0$ and almost every θ,

$$\|\mathcal{C}_\alpha v(\theta)\|_{L^2(\mathbb{R}^3 \setminus B_r)} \lesssim \left((1 + r)^{-1} + \|(1 + |y|)^{-2}\|_{L^3(\mathbb{R}^3 \setminus B_r)}\right) \|v(\theta)\|_{\dot{H}^1}. \tag{7.51}$$

With $r = 0$ this is the bound $\|\mathcal{C}_\alpha v\|_{\mathcal{H}} \le C \|\nabla v\|_{\mathcal{H}}$. If v is Γ-invariant, then so is $\mathcal{C}_\alpha v$, by (7.7) and the properties of the maps ρ_{R,θ_0} stated in Subsection 7.1.

For the compactness, let $f_k \to 0$ weakly in $\mathcal{H}_{\mathrm{div}}$ and $v_k = \mathcal{T}_S^{-1} f_k$. It suffices to prove that $\mathcal{C}_\alpha v_k \to 0$ in $\mathcal{H}$. By (7.43) the sequences (∇v_k) and (Δv_k) are bounded in $\mathcal{H}$, and by Proposition 7.5 with $S_k = S$ we have $v_k \to 0$ in $L^2((0,1) \times B_r)$ for every $r > 0$. Let $r > 0$, and let $\phi \in C_c^\infty(B_{2r})$ be equal to 1 on B_r. Integrating by parts against $\phi^2 v_k(\theta)$, which belongs to

$H^1(\mathbb{R}^3;\mathbb{R}^3)$ for almost every θ, and then over $\theta \in (0,1)$, we get

$$\int_0^1 \int_{\mathbb{R}^3} \phi^2 \, |\nabla v_k|^2 \, dy \, d\theta = - \int_0^1 \int_{\mathbb{R}^3} (\phi^2 \, v_k \cdot \Delta v_k + 2 \, \phi \, (\nabla \phi \cdot \nabla) v_k \cdot v_k) \, dy \, d\theta \\ \lesssim \|v_k\|_{L^2((0,1)\times B_{2r})} (\|\Delta v_k\|_{\mathcal{H}} + \|\nabla v_k\|_{\mathcal{H}}),$$

with a constant that depends only on ϕ, so that $\nabla v_k \to 0$ in $L^2((0,1)\times B_r)$ as well. By (7.50) we have convergence on $(0,1)\times B_r$. The exterior bound (7.51) is $\sup_k \|\nabla v_k\|_{\mathcal{H}}$ times a factor tending to zero as $r \to \infty$, uniformly in k. Choosing r first and then k gives $\mathcal{C}_\alpha v_k \to 0$ in $\mathcal{H}$. □

7.7. The linearized problem. The first step is to pass from the reference profile to the approximate profile in the estimate (7.18). Let $\alpha \in \mathcal{K}$, $v \in \mathcal{D}^\Gamma_{S(\alpha)}$ and $\beta \in \mathbb{R}^n$, and let $F \in \mathcal{H}$ with $F \simeq \mathcal{T}_{S(\alpha)} v + \mathcal{C}_\alpha v + \sum_i \beta_i \, b_i$. Then

$$(1-\vartheta) \, \|(v,\beta)\|_{w,\omega} \le c_1 \, \|F\|_{\mathcal{H}} + [F] + c_2 \, |\ell(v)|. \tag{7.52}$$

Indeed, by (7.13) and (7.16) the field $G = \mathcal{C}_\alpha v - \mathcal{C}^{\mathrm{r}}_\alpha v$ is the field on the left-hand side of (7.19), so that it belongs to $\mathcal{H}$, with $\|G\|_{\mathcal{H}} \le c_{\mathrm{r}} \, \|v\|_w$, and $F - G \simeq \mathcal{T}_{S(\alpha)} v + \mathcal{C}^{\mathrm{r}}_\alpha v + \sum_i \beta_i \, b_i$. We apply (7.18) to the field $F - G$. Since $c_1 \ge 0$, since $[\cdot]$ is a seminorm, and since $[G] \le c_0 \, \|G\|_{\mathcal{H}}$ by (7.17), we obtain

$$\|(v,\beta)\|_{w,\omega} \le c_1 \, \|F-G\|_{\mathcal{H}} + [F-G] + c_2 \, |\ell(v)| \le c_1 \, \|F\|_{\mathcal{H}} + [F] + c_2 \, |\ell(v)| + (c_1 + c_0) \, c_{\mathrm{r}} \, \|v\|_w.$$

By (7.20), the last term is at most $\vartheta \|(v,\beta)\|_{w,\omega}$. Absorbing it gives (7.52), with a positive coefficient because the certificate requires $\vartheta < 1$.

Proposition 7.8. *Let $\alpha \in \mathcal{K}$ and $S = S(\alpha)$. The map*

$$(v,\beta) \mapsto \Big(\mathbb{P}\Big(\mathcal{T}_S v + \mathcal{C}_\alpha v + \sum_{i=1}^n \beta_i b_i \Big), \, \ell(v) \Big) \tag{7.53}$$

is a bijection of $\mathcal{D}^\Gamma_S \times \mathbb{R}^n$ onto $\mathcal{H}^\Gamma_{\mathrm{div}} \times \mathbb{R}^n$. In particular, for every $F \in \mathcal{H}$ with $\mathbb{P}F \in \mathcal{H}^\Gamma_{\mathrm{div}}$ and every prescribed value of $\ell(v)$ in $\mathbb{R}^n$ there is exactly one pair $(v,\beta) \in \mathcal{D}^\Gamma_S \times \mathbb{R}^n$ with $\mathcal{T}_S v + \mathcal{C}_\alpha v + \sum_i \beta_i b_i \simeq F$ and with this value of $\ell(v)$.

Proof. For fields in $\mathcal{H}$ the relation $\simeq$ is the equality of the Leray projections (Subsection 7.1), so that the second assertion follows from the first. Let $(v,\beta) \in \mathcal{D}^\Gamma_S \times \mathbb{R}^n$. The field $\mathcal{T}_S v + \mathcal{C}_\alpha v + \sum_i \beta_i b_i$ belongs to $\mathcal{H}$, by Lemma 7.7, and $\mathcal{T}_S v \in \mathcal{H}^\Gamma_{\mathrm{div}}$, by Proposition 7.5. Hence the first component of (7.53) is

$$\mathcal{T}_S v + \mathbb{P}\mathcal{C}_\alpha v + \sum_{i=1}^n \beta_i \mathbb{P} b_i, \tag{7.54}$$

and it belongs to $\mathcal{H}^\Gamma_{\mathrm{div}}$, because $\mathcal{C}_\alpha v$ and the fields b_i are Γ-invariant and $\mathbb{P}$ maps Γ-invariant fields to Γ-invariant fields.

If the map (7.53) vanishes at (v,β), then $\mathcal{T}_S v + \mathcal{C}_\alpha v + \sum_i \beta_i b_i \simeq 0$ and $\ell(v) = 0$, and (7.52) with $F = 0$ gives $(1-\vartheta) \, \|(v,\beta)\|_{w,\omega} \le 0$. Since $\vartheta < 1$, $w \ge w_* > 0$ and $\omega > 0$, we get $\nabla v = 0$ and $\beta = 0$, and then $v = 0$ by (3.1). Thus the map, which is linear, is injective.

To prove that it is onto we take $f = \mathcal{T}_S v$ as the unknown in place of v, which is possible by Proposition 7.5. On the Hilbert space $\mathcal{H}^\Gamma_{\mathrm{div}} \times \mathbb{R}^n$ let

$$\mathcal{Q}(f,\beta) = \Big(\mathbb{P}\mathcal{C}_\alpha \mathcal{T}_S^{-1} f + \sum_{i=1}^n \beta_i \mathbb{P} b_i, \, \ell(\mathcal{T}_S^{-1} f) - \beta \Big).$$

By (7.54), $I+\mathcal{Q}$ is the composition of the map (7.53) with the bijection $(f,\beta)\mapsto(\mathcal{T}_S^{-1}f,\beta)$ of $\mathcal{H}^\Gamma_{\mathrm{div}}\times\mathbb{R}^n$ onto $\mathcal{D}^\Gamma_S\times\mathbb{R}^n$. The operator $\mathbb{P}\mathcal{C}_\alpha\mathcal{T}_S^{-1}$ is compact, by Lemma 7.7, and the other terms of $\mathcal{Q}$ have finite rank and are bounded, since $|\ell_i(\mathcal{T}_S^{-1}f)|\le\|\Xi_i\|_{\mathcal{H}}\|\nabla\mathcal{T}_S^{-1}f\|_{\mathcal{H}}\le 2\|\Xi_i\|_{\mathcal{H}}\|f\|_{\mathcal{H}}$ by (7.11) and (7.43). Thus $\mathcal{Q}$ is compact on $\mathcal{H}^\Gamma_{\mathrm{div}}\times\mathbb{R}^n$. Estimate (7.52) gives $\ker(I+\mathcal{Q})=\{0\}$. The Fredholm alternative for $-\mathcal{Q}$ now gives surjectivity [14, Theorem 6.6(c), p. 160]. Every Γ-invariant projected forcing and every vector of constraint values therefore determine exactly one pair (v,β). □

Proposition 7.9. *Let $\alpha\in\mathcal{K}$, $v\in\mathcal{D}^\Gamma_{S(\alpha)}$ and $\beta\in\mathbb{R}^n$, and let $F\in\mathcal{H}$ with $F\simeq\mathcal{T}_{S(\alpha)}v+\mathcal{C}_\alpha v+\sum_i\beta_i b_i$. Then*

$$\sup_{\theta\in\mathbb{R}}\|v(\theta)\|_{\dot H^1}^2\le\Big(\frac{1}{w_*}+\frac{2S(\alpha)}{m}\,c_3\Big)\|v\|_w^2+\frac{2S(\alpha)}{m}\big(\|F\|_{\mathcal{H}}+b_*|\beta|\big)\|v\|_w. \tag{7.55}$$

Proof. Set $S=S(\alpha)$ and $\mathcal{E}(\theta)=\|v(\theta)\|_{\dot H^1}^2$. By Lemma 7.4, $\mathcal{E}$ is continuous, locally absolutely continuous and period-1. The relation for F gives $\mathcal{T}_Sv=\mathbb{P}(F-\mathcal{C}_\alpha v-\sum_i\beta_ib_i)$. Since $\Delta v(\theta)\in L^2_{\mathrm{div}}$ almost everywhere, orthogonality removes $\mathbb{P}$ from the pairing in (7.42), giving, for almost every θ,

$$\frac{1}{2S}\mathcal{E}'=-\int_{\mathbb{R}^3}F\cdot\Delta v\,dy+\sum_{i=1}^n\beta_i\int_{\mathbb{R}^3}b_i\cdot\Delta v\,dy-\Big[\|\Delta v\|_{L^2}^2+\tfrac14\mathcal{E}-\int_{\mathbb{R}^3}\mathcal{C}_\alpha v\cdot\Delta v\,dy\Big],$$

where all fields are taken at time θ. The term in brackets is the left-hand side of (7.22) with $u=v(\theta)$. This u is admissible in (7.22): for almost every θ it belongs to X and $\Delta u\in L^2$, and for every $(R,0)\in\Gamma$ the identity $\rho_Rv(\theta)=v(\theta)$ holds for almost every θ, by the Γ-invariance of v, and therefore for every θ, by the continuity of v. By (7.22), for almost every θ,

$$\frac{1}{2S}\mathcal{E}'(\theta)\le\Lambda(\theta),\qquad\Lambda(\theta)=\Big(\|F(\theta)\|_{L^2}+\Big\|\sum_{i=1}^n\beta_ib_i(\theta)\Big\|_{L^2}\Big)\|\Delta v(\theta)\|_{L^2}+c_3|v(\theta)|_w^2.$$

The function Λ is nonnegative, since $c_3\ge0$, and has period 1. By the Cauchy–Schwarz inequality, by (7.9) and since $\|\Delta v\|_{\mathcal{H}}\le\|v\|_w$,

$$\int_0^1\Lambda\,d\theta\le\big(\|F\|_{\mathcal{H}}+b_*|\beta|\big)\|v\|_w+c_3\|v\|_w^2. \tag{7.56}$$

Let θ_1 be a point at which $\mathcal{E}$ attains its minimum. Then

$$\mathcal{E}(\theta_1)\le\int_0^1\mathcal{E}\,d\theta=\|\nabla v\|_{\mathcal{H}}^2\le\frac{1}{w_*}\|v\|_w^2,$$

because $w\ge w_*$. The period of $\mathcal{E}$ is $1/m$. For $m=1$ this is already known. For $m=2$, (7.6) gives R with $(R,\frac12)\in\Gamma$, and the Γ-invariance (7.5) of v gives $v(\theta+\frac12)=\rho_Rv(\theta)$, first almost everywhere and then everywhere by continuity; ρ_R preserves the X norm. Thus, for $\theta\in[\theta_1,\theta_1+\frac1m]$,

$$m(\mathcal{E}(\theta)-\mathcal{E}(\theta_1))=\sum_{k=0}^{m-1}\Big[\mathcal{E}(\theta+\tfrac{k}{m})-\mathcal{E}(\theta_1+\tfrac{k}{m})\Big]\le2S\sum_{k=0}^{m-1}\int_{\theta_1+k/m}^{\theta+k/m}\Lambda\,d\theta'\le2S\int_0^1\Lambda\,d\theta',$$

because the intervals have disjoint interiors, $\Lambda\ge0$, and its integral over any interval of length 1 is the same. Periodicity of $\mathcal{E}$ then gives

$$\sup_{\theta\in\mathbb{R}}\mathcal{E}(\theta)\le\frac{1}{w_*}\|v\|_w^2+\frac{2S}{m}\int_0^1\Lambda\,d\theta,$$

and (7.55) follows from (7.56). □

7.8. The nonlinear problem and the choice of the parameter. We seek a parameter $\alpha \in \mathcal{K}$ and a solution (v, β) of (7.12) for which $\beta^{\mathrm{a}}(\alpha) + \beta = 0$. Regard the quadratic term provisionally as a field $g \in \mathcal{H}$ so that the problem at (α, g) is linear. For fixed $\alpha \in \mathcal{K}$ and g we consider

$$\mathcal{T}_{S(\alpha)} v + \mathcal{C}_\alpha v + \sum_{i=1}^{n} \beta_i b_i \simeq -F^{\mathrm{a}}_\alpha - g, \qquad \ell(v) = \bar{\ell}(\alpha), \tag{7.57}$$

for $(v, \beta) \in \mathcal{D}^\Gamma_{S(\alpha)} \times \mathbb{R}^n$. Such a pair solves (7.12) at α exactly when it solves (7.57) with $g = (v \cdot \nabla) v$.

Updating g and α together avoids choosing nonlinear solutions continuously in α. Only the linear solution at (α, g) must be unique.

The residual enters (7.57) through two consequences of the relation (7.10):

$$\mathbb{P} F^{\mathrm{a}}_\alpha \in \mathcal{H}^\Gamma_{\mathrm{div}} \quad (\alpha \in \mathcal{K}), \qquad \alpha \mapsto \int_{\mathbb{R}} \int_{\mathbb{R}^3} F^{\mathrm{a}}_\alpha \cdot \varphi \, dy \, d\theta \text{ is continuous on } \mathcal{K}, \tag{7.58}$$

where continuity is tested against each divergence-free $\varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$. For $(R, \theta_0) \in \Gamma$, (7.7) and the invariance of the b_i make (7.10) invariant modulo gradients under ρ_{R,θ_0}. Preservation of $\simeq$ gives $\rho_{R,\theta_0} F^{\mathrm{a}}_\alpha \simeq F^{\mathrm{a}}_\alpha$. Both fields lie in $\mathcal{H}$ and $\mathbb{P}$ commutes with the action, so $\rho_{R,\theta_0} \mathbb{P} F^{\mathrm{a}}_\alpha = \mathbb{P} \rho_{R,\theta_0} F^{\mathrm{a}}_\alpha = \mathbb{P} F^{\mathrm{a}}_\alpha$. For continuity, substitute (7.10) and integrate the Laplacian by parts. On the compact support of φ, the factors W^{a}_α, $\partial_\theta W^{\mathrm{a}}_\alpha$ and $\nabla W^{\mathrm{a}}_\alpha$ vary uniformly by joint continuity on its product with $\mathcal{K}$. The coefficients $1/S(\alpha)$ and $\beta^{\mathrm{a}}_i(\alpha)$ are continuous since S is positive, and the pairings of b_i with φ are fixed.

Given (α, g), solve (7.57) for (v, β) and form $(v \cdot \nabla) v$. This is the new value of g. For the new parameter take the nearest point of $\mathcal{K}$ to α' from (7.30). At a Schauder–Tychonoff fixed point, $g = (v \cdot \nabla) v$. The bounds (7.61) put the proposed α' in $\mathcal{K}$. Projection then fixes α', and (7.30) cancels the total force.

Proposition 7.10. *There exist $\alpha \in \mathcal{K}$ and a pair $(v, \beta) \in \mathcal{D}^\Gamma_{S(\alpha)} \times \mathbb{R}^n$ which solves (7.12) at α and satisfies*

$$\|(v, \beta)\|_{w,\omega} \le 2\varepsilon, \qquad \sup_{\theta \in \mathbb{R}} \|v(\theta)\|_{\dot{H}^1} \le M \tag{7.59}$$

and

$$\beta^{\mathrm{a}}(\alpha) + \beta = 0. \tag{7.60}$$

Proof. Let $\mathcal{Z}$ be the set of Γ-invariant fields $g \in \mathcal{H}$ with $\|g\|_{\mathcal{H}} \le 2 c_w \varepsilon M$. Let $\alpha \in \mathcal{K}$, $S = S(\alpha)$ and $g \in \mathcal{Z}$. The field $F = -F^{\mathrm{a}}_\alpha - g$ belongs to $\mathcal{H}$, and $\mathbb{P} F \in \mathcal{H}^\Gamma_{\mathrm{div}}$, by (7.58) and because $\mathbb{P}$ preserves Γ-invariance (Subsection 7.1). By Proposition 7.8 there is exactly one pair (v, β) in $\mathcal{D}^\Gamma_S \times \mathbb{R}^n$ which satisfies (7.57); we call it the solution at (α, g).

We check that the solution at (α, g) obeys (7.59), which will also put $(v \cdot \nabla) v$ in $\mathcal{Z}$. The residual bounds (7.23), the definition of $\mathcal{Z}$ and (7.17) give

$$\|F\|_{\mathcal{H}} \le \delta(\alpha) + 2 c_w \varepsilon M, \qquad [F] \le \delta_0(\alpha) + 2 c_0 c_w \varepsilon M.$$

Apply the linear estimate (7.52), then the residual inequality (7.24) and the second inequality of (7.21):

$$\begin{aligned} (1 - \vartheta) \, \|(v, \beta)\|_{w,\omega} &\le c_1 \delta(\alpha) + \delta_0(\alpha) + c_2 |\bar{\ell}(\alpha)| + 2(c_1 + c_0) c_w \varepsilon M \\ &\le (1 - \vartheta)\varepsilon + 2(1 - \vartheta) c_L c_w \varepsilon M. \end{aligned}$$

Since $\vartheta < 1$, division by $1 - \vartheta$ and the smallness condition (7.25) yield

$$\|(v, \beta)\|_{w,\omega} \le \varepsilon (1 + 2 c_L c_w M) \le 2\varepsilon.$$

We have $\|v\|_w \le 2\varepsilon$, $|\beta| \le 2\varepsilon/\omega$, $\|F\|_{\mathcal{H}} \le \delta_{\max} + 2c_w\varepsilon M$ by (7.23), and $S \le S_{\max}$. Inserting these in (7.55) and using (7.27) bounds $\sup_\theta \|v(\theta)\|_{\dot H^1}^2$ by M^2. Since $M > 0$, this proves (7.59). Lemma 7.6 gives $\|(v\cdot\nabla)v\|_{\mathcal{H}} \le 2c_w\varepsilon M$, and Γ-invariance of v puts the product in $\mathcal{Z}$.

Let $\mathcal{N}$ be the nearest-point projection, for the Euclidean norm, onto the nonempty closed convex set $\mathcal{K}$. It is continuous and fixes every point of $\mathcal{K}$ [14, Theorem 5.2, p. 132; Proposition 5.3, p. 134]. For $(\alpha, g) \in \mathcal{K}\times\mathcal{Z}$ define

$$\mathcal{G}(\alpha, g) = (\mathcal{N}(\alpha'),\, (v\cdot\nabla)v),$$

where (v,β) solves (7.57) at (α, g) and α' is given by (7.30). The first component lies in $\mathcal{K}$ by definition of the projection, and the second lies in $\mathcal{Z}$ by the preceding bounds. Thus $\mathcal{G}$ maps $\mathcal{K}\times\mathcal{Z}$ into itself.

Give $\mathcal{K}$ its Euclidean topology and $\mathcal{Z}$ its weak topology. The latter is a closed ball in the closed subspace of Γ-invariant fields in the separable Hilbert space $\mathcal{H}$. Hence $\mathcal{Z}$ is weakly compact [14, Corollary 3.22, p. 71] and metrizable [14, Theorem 3.29, p. 74]. The product $\mathcal{K}\times\mathcal{Z}$ is compact, convex and nonempty in $\mathbb{R}^n\times\mathcal{H}$ with $\mathcal{H}$ weak. Schauder–Tychonoff [88, §2, p. 770] gives a fixed point once $\mathcal{G}$ is sequentially continuous. Let $\alpha_k \to \alpha$ and $g_k \rightharpoonup g$ in $\mathcal{H}$ with pairs in $\mathcal{K}\times\mathcal{Z}$ and corresponding solutions (v_k,β_k) and (v,β). By continuity of β^{a} and $\mathcal{N}$ it suffices to show $\beta_k \to \beta$ and $(v_k\cdot\nabla)v_k \rightharpoonup (v\cdot\nabla)v$ in $\mathcal{H}$.

It suffices to extract from each subsequence a further subsequence with these same limits. Put $S_k = S(\alpha_k)$, $S = S(\alpha)$ and $f_k = \mathcal{T}_{S_k} v_k$. Lemma 7.4 puts f_k in $\mathcal{H}_{\mathrm{div}}$, so projection of (7.57) gives

$$f_k = -\mathbb{P}\Big(F^{\mathrm{a}}_{\alpha_k} + g_k + \mathcal{C}_{\alpha_k} v_k + \sum_{i=1}^n \beta_{k,i} b_i\Big).$$

The right side is bounded in $\mathcal{H}$ by (7.23), membership of g_k in $\mathcal{Z}$, Lemma 7.7 with $\|\nabla v_k\|_{\mathcal{H}} \le w_*^{-1/2}\|v_k\|_w$, and (7.9), since (7.59) controls $\|v_k\|_w$ and $|\beta_k|$. From every subsequence extract $f_k \rightharpoonup f$ in $\mathcal{H}_{\mathrm{div}}$ and $\beta_k \to \tilde\beta$ in $\mathbb{R}^n$. Weak closure of the invariant subspace gives $f \in \mathcal{H}^\Gamma_{\mathrm{div}}$ and $\tilde v = \mathcal{T}_S^{-1} f \in \mathcal{D}^\Gamma_S$. Continuity of S gives $S_k \to S > 0$. Proposition 7.5 gives $\nabla v_k \rightharpoonup \nabla\tilde v$ in $\mathcal{H}$ and $v_k \to \tilde v$ in $L^2((0,1)\times B_r)$ for every $r > 0$. Periodicity extends the strong limit to bounded space-time sets.

We identify the limiting pair by the equation and constraints. For a divergence-free $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$, (7.57) gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3}\Big[f_k + (W^{\mathrm{a}}_{\alpha_k}\cdot\nabla)v_k + (v_k\cdot\nabla)W^{\mathrm{a}}_{\alpha_k} + \sum_{i=1}^n \beta_{k,i} b_i + F^{\mathrm{a}}_{\alpha_k} + g_k\Big]\cdot\varphi\, dy\, d\theta = 0,$$

Periodise φ as in (7.3). The terms f_k and g_k converge weakly in $\mathcal{H}$. On the support of φ, $W^{\mathrm{a}}_{\alpha_k}$ and $\nabla W^{\mathrm{a}}_{\alpha_k}$ converge uniformly to W^{a}_α and $\nabla W^{\mathrm{a}}_\alpha$ by joint continuity, so the weak gradient and strong velocity limits pass both convection terms. The force converges by $\beta_k \to \tilde\beta$, the residual pairing by (7.58). With $f = \mathcal{T}_S\tilde v$ this gives the equation in (7.57). The constraint follows from $\ell(v_k) \to \ell(\tilde v)$ by weak gradient convergence and $\bar\ell(\alpha_k) \to \bar\ell(\alpha)$ by continuity. Thus $(\tilde v, \tilde\beta) = (v,\beta)$ by Proposition 7.8. For the quadratic term, test against any compactly supported smooth φ and split the difference as $((v_k - v)\cdot\nabla)v_k + (v\cdot\nabla)(v_k - v)$. Strong local L^2 convergence and bounded gradients control the first pairing, weak gradient convergence the second. The periodised tests $\sum_{l\in\mathbb{Z}}\varphi(\theta + l, y)$ are dense in $\mathcal{H}$ (§7.1), and the bound from $\mathcal{Z}$ extends convergence to all of $\mathcal{H}$. Hence $(v_k\cdot\nabla)v_k \rightharpoonup (v\cdot\nabla)v$ along this subsequence. The subsequence criterion gives continuity of $\mathcal{G}$.

Choose a fixed point (α, g) and its solution (v,β). Its second component gives $g = (v\cdot\nabla)v$. Thus $(v,\beta) \in \mathcal{D}^\Gamma_{S(\alpha)} \times \mathbb{R}^n$ solves (7.12) and satisfies (7.59).

For the force cancellation (7.60), the first component gives $\alpha = \mathcal{N}(\alpha')$, where α' is defined by (7.30). We show that α' already lies in $\mathcal{K}$. Conditions (7.32)–(7.37) give

$$\begin{aligned} &|V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j \alpha_j| \le |d_j|\, \varrho_j \qquad \text{for } j \in \mathcal{J}, \\ &\sum_{j \notin \mathcal{J}} \big(V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j \alpha_j\big)^2 \le \varrho_*^2. \end{aligned} \tag{7.61}$$

Indeed, $d_j \alpha_j' = -(V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j \alpha_j)$. Since every $d_j \neq 0$, (7.61) is exactly the set of inequalities defining $\alpha' \in \mathcal{K}$ in (7.29). The projection therefore fixes α', so $\alpha = \alpha'$. Equation (7.30) now gives $V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) = 0$ for every j. The columns of the invertible matrix V span $\mathbb{R}^n$, proving (7.60).

To prove (7.61), let $j \in \mathcal{J}$. The field

$$F' = -F_\alpha^{\mathrm{a}} - (v \cdot \nabla) v - \sum_{i=1}^{n} \beta_i b_i$$

lies in $\mathcal{H}$ by Lemma 7.6 and satisfies $F' \simeq \mathcal{T}_{S(\alpha)} v + \mathcal{C}_\alpha v$ by (7.12). Substitute it into (7.33), use $\ell(v) = \bar{\ell}(\alpha)$ and multiply the expression inside the absolute value by -1. With (7.31), this gives

$$\big|\langle F_\alpha^{\mathrm{a}}, \psi_{\alpha,j} \rangle_{\mathcal{H}} + \langle (v \cdot \nabla) v, \psi_{\alpha,j} \rangle_{\mathcal{H}} + \beta \cdot \langle b, \psi_{\alpha,j} \rangle_{\mathcal{H}} + \big(Y(\alpha)\, \bar{\ell}(\alpha)\big)_j \big| \le a_j\, \|v\|_w.$$

Split $\beta \cdot \langle b, \psi_{\alpha,j} \rangle_{\mathcal{H}}$ as $V_j \cdot \beta + \beta \cdot (\langle b, \psi_{\alpha,j} \rangle_{\mathcal{H}} - V_j)$. Bound the quadratic term by (7.32). Cauchy–Schwarz in $\mathbb{R}^n$ bounds the coefficient error; Cauchy–Schwarz in $\mathbb{R}^2$ combines it with $a_j \|v\|_w$ using the two components $\|v\|_w$ and $\omega|\beta|$. Applying (7.59) yields

$$\begin{aligned} |V_j \cdot \beta + \langle F_\alpha^{\mathrm{a}}, \psi_{\alpha,j} \rangle_{\mathcal{H}} + \big(Y(\alpha)\, \bar{\ell}(\alpha)\big)_j| &\le a_j\, \|v\|_w + \frac{|\langle b, \psi_{\alpha,j} \rangle_{\mathcal{H}} - V_j|}{\omega}\, \omega\, |\beta| + \tilde{a}_j\, \|v\|_w^2 \\ &\le \Big(a_j^2 + \frac{|\langle b, \psi_{\alpha,j} \rangle_{\mathcal{H}} - V_j|^2}{\omega^2}\Big)^{1/2} \|(v, \beta)\|_{w,\omega} + \tilde{a}_j\, \|v\|_w^2 \\ &\le 2\varepsilon \Big(a_j^2 + \frac{|\langle b, \psi_{\alpha,j} \rangle_{\mathcal{H}} - V_j|^2}{\omega^2}\Big)^{1/2} + \tilde{a}_j\, (2\varepsilon)^2. \end{aligned}$$

Now

$$\begin{aligned} V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j \alpha_j = &\Big[V_j \cdot \beta^{\mathrm{a}}(\alpha) - \langle F_\alpha^{\mathrm{a}}, \psi_{\alpha,j} \rangle_{\mathcal{H}} - \big(Y(\alpha)\, \bar{\ell}(\alpha)\big)_j - d_j \alpha_j\Big] \\ &+ \Big[V_j \cdot \beta + \langle F_\alpha^{\mathrm{a}}, \psi_{\alpha,j} \rangle_{\mathcal{H}} + \big(Y(\alpha)\, \bar{\ell}(\alpha)\big)_j\Big]. \end{aligned}$$

The first bracket has absolute value at most o_j by (7.34), and the second has the bound just obtained. Their sum is bounded by the left side of (7.35), hence by $|d_j| \varrho_j$. This is the first inequality of (7.61).

For $j \notin \mathcal{J}$, split $V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j \alpha_j$ into $V_j \cdot \beta$ and $V_j \cdot \beta^{\mathrm{a}}(\alpha) - d_j \alpha_j$. Take the Euclidean norm of this vector of complementary coordinates. The two bounds in (7.36), then $\omega|\beta| \le 2\varepsilon$ from (7.59), and finally (7.37), give

$$\Big(\sum_{j \notin \mathcal{J}} \big(V_j \cdot (\beta^{\mathrm{a}}(\alpha) + \beta) - d_j \alpha_j\big)^2\Big)^{1/2} \le a_*\, |\beta| + o_* \le \frac{2\varepsilon\, a_*}{\omega} + o_* \le \varrho_*.$$

□

7.9. Proof of Theorem 7.1. By Proposition 7.10 the force vanishes at the selected parameter. The estimate on the correction gives a nonzero field once we use (7.28). It remains to recover smoothness and the pointwise decay (1.3) from its weak equation. For this we use Proposition 7.11.

The correction norm alone gives no pointwise decay. For the sum $W_\alpha^{\mathrm{a}} + v$, (7.63)–(7.65) improve the Eulerian L^6 estimate to L^∞. Bounded mild-solution regularity [69, Proposition 4.1] gives smoothness before $t = 1$, and the exterior estimate (7.66) gives spatial decay.

Proposition 7.11. *Let $S > 0$, and let $\mathcal{W} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ be a field such that the map $\theta \mapsto \mathcal{W}(\theta)$ is continuous from $\mathbb{R}$ to X, has period 1 and is not identically zero. Suppose that every term of $\mathcal{T}_S\mathcal{W} + (\mathcal{W}\cdot\nabla)\mathcal{W}$ is a locally integrable function on $\mathbb{R}\times\mathbb{R}^3$, with $\mathcal{T}_S$ as in (7.1), and that*

$$\mathcal{T}_S\mathcal{W} + (\mathcal{W}\cdot\nabla)\mathcal{W} \simeq 0$$

in the sense of (7.2). Then there exist smooth functions $W : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ and $q : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ such that (W, q) is a profile with period S and isometry I and

$$W(s,\cdot) = \mathcal{W}\left(\tfrac{s}{S}\right) \qquad \text{in } X \quad \text{for every } s \in \mathbb{R}. \tag{7.62}$$

Proof of Theorem 7.1. Take $\alpha, (v,\beta)$ from Proposition 7.10 and set $S = S(\alpha)$ and $\mathcal{W} = W_\alpha^{\mathrm{a}} + v$. We check Proposition 7.11. Lemma 7.4 gives continuity and period 1 for v as an X-valued map. For W_α^{a}, continuity of its gradient and (7.8) give the same conclusion by dominated convergence: the squared gradient is bounded by a constant times the integrable function $(1+|y|)^{-4}$. Hence $\mathcal{W}$ is continuous into X and has period 1.

With (7.60), the relation after (7.13) becomes $\mathcal{T}_S\mathcal{W} + (\mathcal{W}\cdot\nabla)\mathcal{W} \simeq 0$. Its terms are locally integrable: the other terms follow from the conditions of Subsection 7.2 on W_α^{a}, from $v \in \mathcal{D}_S$ and from Lemmas 7.6 and 7.7; for $\partial_\theta v$, use (7.4), since v and $(y\cdot\nabla)v$ are locally square integrable. To prove nonvanishing, suppose $\mathcal{W} = 0$. Then $\nabla W_\alpha^{\mathrm{a}} = -\nabla v$, and $w \geq w_* > 0$ together with (7.59) gives

$$w_* \left\|\nabla W_\alpha^{\mathrm{a}}\right\|_{\mathcal{H}}^2 = w_* \left\|\nabla v\right\|_{\mathcal{H}}^2 \leq \|v\|_w^2 \leq 4\varepsilon^2,$$

contrary to the strict lower bound (7.28).

Proposition 7.11 gives (W, q) with period S, isometry I and $W(S\theta,\cdot) = \mathcal{W}(\theta)$ in X for every θ. Subtracting W_α^{a} recovers $v \in \mathcal{D}_S^\Gamma$. Since $\beta = -\beta^{\mathrm{a}}(\alpha)$, (7.59) gives the two bounds in (7.38), and (7.12) gives the constraint.

For (7.39), fix $(R,\theta_0) \in \Gamma$. Equation (7.7) gives $W_\alpha^{\mathrm{a}}(\theta+\theta_0) = \rho_R W_\alpha^{\mathrm{a}}(\theta)$ at every θ. Invariance of v gives the same equality almost everywhere in θ by (7.5). Its two sides are continuous into X, so equality extends to every θ. Adding gives $\mathcal{W}(\theta+\theta_0) = \rho_R\mathcal{W}(\theta)$ and hence $W(s+\theta_0 S,\cdot) = \rho_R W(s,\cdot)$ in X for every s. Equality in X is equality almost everywhere in space; spatial continuity of W extends it to every y, proving (7.39). □

7.10. Proof of Proposition 7.11. Let S and $\mathcal{W}$ be as in the proposition, and let u be the field (7.40) with $\mathcal{W}$ in place of v: for $t < 1$ we have $u(t) = D_{1/(1-t)}\mathcal{W}(\theta)$ with $\theta = -\frac{1}{S}\log(1-t)$. By the continuity of the scaling (Subsection 3.1), u is a continuous map from $(-\infty, 1)$ to X. By a change of variables and by (3.1),

$$\sup_{t<1}(1-t)^{1/4}\|u(t)\|_{L^6} = \sup_{\theta\in\mathbb{R}}\|\mathcal{W}(\theta)\|_{L^6} < \infty, \tag{7.63}$$

because $\mathcal{W}$ is continuous and periodic. In this subsection the constants implied by the symbol $\lesssim$ depend only on the number (7.63).

We first prove the mild identity on every $[a,b]$ with $a < b < 1$. The local integrability assumptions on $\mathcal{W}$, the chain rule (7.41) and $(u\cdot\nabla)u(t,x) = e^{3S\theta/2}(\mathcal{W}\cdot\nabla)\mathcal{W}(\theta,y)$ give local

integrability of $\partial_t u$, Δu and $(u\cdot\nabla)u$. For $\phi\in C_c^\infty((-\infty,1))$ and $\psi\in C^\infty_{c,\mathrm{div}}$, the transformed test $e^{-S\theta}\phi(t)\psi(x)$ is smooth, divergence-free and compactly supported: the time support stays away from $t=1$. Since $dx\,dt = Se^{-5S\theta/2}\,dy\,d\theta$, testing the weak equation gives

$$\begin{aligned}\int_{-\infty}^1 \phi(t)\int_{\mathbb{R}^3}\big[\partial_t u-\Delta u+(u\cdot\nabla)u\big]\cdot\psi\,dx\,dt\\ = S\int_{\mathbb{R}}\int_{\mathbb{R}^3}\big[\mathcal{T}_S\mathcal{W}+(\mathcal{W}\cdot\nabla)\mathcal{W}\big](\theta,y)\cdot e^{-S\theta}\phi(t)\,\psi(x)\,dy\,d\theta=0.\end{aligned}$$

Since $u(t)$ is divergence-free, an integration by parts on the left-hand side gives

$$\int_{-\infty}^1\phi'(t)\int_{\mathbb{R}^3}u(t)\cdot\psi\,dx\,dt=-\int_{-\infty}^1\phi(t)\int_{\mathbb{R}^3}\Big[u(t)\cdot\Delta\psi+\big(u(t)\otimes u(t)\big):\nabla\psi\Big]\,dx\,dt.$$

Both time-dependent pairings are continuous, since u is continuous into $X\hookrightarrow L^6$ and ψ has compact support. The identity says that the distributional derivative of the left pairing is the continuous right pairing in (3.14). Subtracting an antiderivative leaves a distribution with zero derivative, hence a constant. The original continuous pairing is therefore continuously differentiable and satisfies (3.14) at every $t<1$. Lemma 3.6 gives the mild identity on $[a,b]$.

We next prove $\|u(t)\|_{L^\infty}\lesssim(1-t)^{-1/2}$ for each $t<1$. By (2.4) and the second identity of (3.10), in which O_τ is the kernel of $e^{\tau\Delta}\mathbb{P}$, for all $a<t<1$

$$u(t)=e^{(t-a)\Delta}u(a)-\int_a^t\nabla O_{t-t'}*(u(t')\otimes u(t'))\,dt',\tag{7.64}$$

where the integral converges in X, hence in L^6. A direct L^∞ estimate from $u\otimes u\in L^3$ would use $\|\nabla O_\tau\|_{3/2}\lesssim\tau^{-1}$, which is not integrable at zero. Passing first to L^{12} gives the two integrable powers $\tau^{-7/8}$ and $\tau^{-3/4}$. Fix $t<1$ and choose $a=t-(1-t)$. On $[a,t]$ we have $1-t\le 1-t'\le 2(1-t)$, so (7.63) gives $\|u(t')\|_{L^6}\lesssim(1-t)^{-1/4}$. The heat kernel in (2.1) has $L^{12/11}$ norm proportional to $\tau^{-1/8}$, while (3.9) gives $\|\nabla O_\tau\|_{L^{4/3}}\lesssim\tau^{-7/8}$. Apply Young's convolution inequality to $u(a)$ and to $u\otimes u\in L^3$, then Minkowski's inequality to the time integral:

$$\|u(t)\|_{L^{12}}\lesssim(1-t)^{-1/8}\|u(a)\|_{L^6}+\int_a^t(t-t')^{-7/8}\|u(t')\|_{L^6}^2\,dt'\lesssim(1-t)^{-3/8}.$$

The last estimate uses $\int_0^{1-t}\tau^{-7/8}\,d\tau=8(1-t)^{1/8}$. Repeat the argument using this L^{12} bound, valid at every time, and $u\otimes u\in L^6$. Now (3.9) gives $\|\nabla O_\tau\|_{L^{6/5}}\lesssim\tau^{-3/4}$, and $\int_0^{1-t}\tau^{-3/4}\,d\tau=4(1-t)^{1/4}$:

$$\|u(t)\|_{L^\infty}\lesssim(1-t)^{-1/8}\|u(a)\|_{L^{12}}+\int_a^t(t-t')^{-3/4}\|u(t')\|_{L^{12}}^2\,dt'\lesssim(1-t)^{-1/2}.\tag{7.65}$$

In particular u is bounded on $(-\infty,b]\times\mathbb{R}^3$ for every $b<1$.

Fix $a<b<1$. By (7.65) and $\|\nabla O_\tau\|_{L^1}\lesssim\tau^{-1/2}$ from (3.9), (7.64) converges absolutely pointwise and Fubini identifies it almost everywhere with the X-valued integral. Translating a to zero gives the Stokes formula [69, (3.5), preprint p. 7] with datum $u(a)$, force $-\sum_m\partial_m(u_m u)$ and kernel $\partial_m(O_\tau)_{ij}$. Thus $u(a+t,x)$ is a bounded mild solution with bounded datum [69, Section 4, preprint p. 10]. Proposition 4.1 [69, preprint p. 11] bounds the distributional derivatives $(t-a)^{k/2+l}\nabla_x^k\partial_t^l u$ on $(a,b)\times\mathbb{R}^3$ for all integers $k,l\ge0$. Starting earlier gives bounded derivatives of every order on each compact interval, hence a smooth representative of u on $(-\infty,1)\times\mathbb{R}^3$. Its slices represent $u(t)$ in X for every t by continuity from almost every time.

Set $G=\partial_t u-\Delta u+(u\cdot\nabla)u$, which is smooth. Equation (3.14) gives $\int_{\mathbb{R}^3}G(t)\cdot\psi\,dx=0$ for every compactly supported solenoidal test ψ and every $t<1$. Taking ψ to be the curl of a

compactly supported smooth field gives $\operatorname{curl} G(t) = 0$. The radial primitive on $\mathbb{R}^3$ then gives $G = -\nabla p$ with

$$p(t,x) = -\int_0^1 G(t,\sigma x)\cdot x\, d\sigma,$$

which is smooth in (t,x). To check the gradient, use $\partial_i G_j = \partial_j G_i$ and differentiate under the integral: the resulting integrand is $\partial_\sigma(\sigma G_i(t,\sigma x))$. Thus $\partial_i p = -G_i$, and (u,p) satisfies (1.1) throughout $(-\infty,1)\times\mathbb{R}^3$.

Since $\mathcal{W}$ has period 1, the second identity of (3.5) gives $u(1-e^{-S}(1-t)) = D_{e^S}u(t)$ for $t<1$, which is the relation (1.5) with $T=1$, $R=I$ and $\lambda = e^{-S}$. Both sides are continuous functions on $\mathbb{R}^3$, and with $1+e^S t$ in place of t we obtain

$$u(1+t,x) = e^{S/2}\,u(1+e^S t, e^{S/2}x) \qquad (t<0,\ x\in\mathbb{R}^3).$$

Thus $(x,t)\mapsto u(1+t,x)$ is discretely self-similar on $\mathbb{R}^3\times(-\infty,0)$ with factor $e^{S/2}>1$, as defined in [26, (1.2), preprint p. 2]. It is smooth, continuous in time with values in $L^6(\mathbb{R}^3)$ by (3.1), and solves (1.1) with pressure $p(1+t,x)$. Chae and Wolf's Theorem 1.1 applies with exponent 6 and this factor [26, Theorem 1.1, preprint p. 2]. It supplies a constant C such that

$$|u(t,x)| \le \frac{C}{\sqrt{1-t}+|x|} \qquad (t<1,\ x\in\mathbb{R}^3). \tag{7.66}$$

It remains to return to the self-similar variables. For $(s,y)\in\mathbb{R}\times\mathbb{R}^3$ let

$$W(s,y) = e^{-s/2}\,u(1-e^{-s}, e^{-s/2}y), \qquad q(s,y) = e^{-s}\,p(1-e^{-s}, e^{-s/2}y).$$

These functions are smooth, and $W(s,\cdot) = D_{e^{-s}}u(1-e^{-s}) = \mathcal{W}(s/S)$ for every s, which is (7.62). In particular W is not identically zero, and $W(s+S,\cdot) = W(s,\cdot)$ in X for every s, hence at every point, because W is continuous; this is the third equation of (1.2) with $R=I$. The variables (s,y) and (t,x) are related as in (7.40) with $S=1$, and by the chain rule (7.41) the equations (1.1) for (u,p) are the first two equations of (1.2) for (W,q). Finally, with $t=1-e^{-s}$ and $x=e^{-s/2}y$ the bound (7.66) reads $|W(s,y)|\le C\,(1+|y|)^{-1}$ for all $(s,y)\in\mathbb{R}\times\mathbb{R}^3$, which is the first bound of (1.3). By Remark 1.2, the second bound follows from the first and from the first two equations of (1.2), with a constant that depends only on C. Thus (W,q) is a profile with period S and isometry I. □

Related constructions. For time-periodic Navier–Stokes solutions with a prescribed, time-independent force, [90, (2.8)–(2.14), pp. 7–9 of the preprint] treats the frequency as an unknown in a space-time Fourier expansion and fixes the phase. In [9, (2.14)–(2.15), p. 5] the phase condition is imposed after rescaling time. Discretely self-similar constructions [80, Section 1.1, p. 5 of the preprint], [22, Section 1.3.1, pp. 4–5] and the general formulation [54, Section 3.3, (16)–(17), p. 8 of the preprint] likewise find the orbit and its frequency or period together near an approximation. Symmetry reduces the unknowns, with sixteen elements in [90, p. 3, and §5.2, (5.6), p. 43 of the preprint] and a half-period sign change in [22, Section 1.3.1, p. 4]. For the data in Appendix B with $k_{\mathrm{rot}} = 19$, a half-period time shift gives a rotation by $\pi/19$ about one axis. Rotation by π about a perpendicular axis is a purely spatial symmetry. The constraints include a choice of time phase, while Proposition 7.10 selects $S(\alpha)$ together with the correction.

For forced flow on the three-torus, [90, Theorem 1.1, preprint p. 3] identifies the Fourier data file and bounds the frequency, velocity and pressure errors after matching time scales. For planar flow, [9, §4.4, pp. 11–13] describes the Fourier polynomial, enclosures for its remainder and the program that checks the estimates.

For Swift–Hohenberg patterns on a two-dimensional torus, [89, (2.3), p. 4] also adds a force in a fixed finite span and the same number of constraints. The forcing fields are translation derivatives of the approximate solution. Translation invariance of the energy makes pairing with derivatives of the exact solution a homogeneous linear system for the forcing coefficients; Theorem 2.6, p. 7, of [89] sets those coefficients to zero once its matrix is invertible. Here the pairings with $\psi_{\alpha,j}$ give the bounds following (7.30). For the approximate family described in Appendix C, one forcing field accounts for time translations and the others are computed directions in which the linearized inverse is largest (Subsection C.4). Donninger and Schörkhuber [44, Section 2.3, steps (5)–(8), pp. 8–9 of the preprint] use one parameter for cubic Schrödinger blow-up: they regularize a formal inverse by adding a term along a fixed function, solve the modified equation by contraction at each small parameter, and then use the intermediate value theorem to make its added coefficient vanish.

The inverse estimate (7.18) has several stationary precedents. For Navier–Stokes in forward self-similar variables on $\mathbb{R}^3$, [62, Section 3.2, Assumptions 2–3, pp. 19–20] splits the operator into coercive and compact parts and approximates the latter by finite rank, following the strategy of Chen and Hou [32, Section 2.2, steps 5–6, p. 9 of the arXiv preprint]. For flow on an unbounded strip with an obstacle, [95, Section 3.2, pp. 33–35; Section 6.2, (6.7)–(6.10), pp. 68–69, and Theorems 6.7–6.8, pp. 71–72] obtains inverse bounds from spectral enclosures. The construction in [23, Section 3.2, (26)–(28), p. 13 of the preprint] instead uses an approximate inverse represented by Fourier series on a box.

Appendix A. Proofs of the lemmas of Sections 2 and 3

A.1. Proofs of the lemmas of Section 3.

Proof of Lemma 3.3. Bilinearity, continuity in τ and (3.12) make $(\tau,\chi,\chi') \mapsto K(\tau)(\chi,\chi')$ jointly continuous for $\tau > 0$. For $a \le t' < t$ the integrand is continuous and bounded by $C_K(t-t')^{-3/4}\max_{[a,b]}\|u\|_{\mathcal{X}}\max_{[a,b]}\|v\|_{\mathcal{X}}$, integrable since $3/4 < 1$. For continuity in t, extend u, v by their endpoint values, put $\tau = t - t'$ and integrate over $(0, b-a)$ with zero integrand for $\tau > t - a$. For almost every τ the integrand is continuous at t under the same bound. Dominated convergence includes $t = a$. □

Proof of Lemma 3.4. For $\beta > 0$, equip $C([a,b];\mathcal{X})$ with $\|h\|_\beta = \max_{a\le t\le b} e^{-\beta(t-a)}\|h(t)\|_{\mathcal{X}}$, an equivalent complete norm. Let Lh denote the integral term in the lemma. Lemma 3.3 gives $Lh \in C([a,b];\mathcal{X})$. By (3.12) and $e^{-\beta(t-a)} = e^{-\beta(t-t')}e^{-\beta(t'-a)}$, its operator norm is at most

$$2C_K \max_{[a,b]}\|f\|_{\mathcal{X}} \int_0^\infty \tau^{-3/4}e^{-\beta\tau}\,d\tau,$$

where the substitution $s = \beta\tau$ makes the integral a finite constant times $\beta^{-1/4}$. Choose β so that $\|L\|_\beta < 1$. Then $h \mapsto Lh + g$ has a unique fixed point, and the convergent series $(I-L)^{-1} = \sum_{j\ge0} L^j$ shows that $g \mapsto h$ is linear and bounded. Equivalence of norms gives the asserted supremum-norm bound. □

Proof of Lemma 3.5. Apply $e^{(t-m)\Delta}$ to the identity at time m. This bounded linear operator commutes with the Bochner integral, which is integrable by Lemma 3.3. The semigroup law (3.11) and the first identity in (3.12) give

$$e^{(t-m)\Delta}u(m) = e^{(t-a)\Delta}u(a) + \int_a^m K(t-t')(f(t'),h(t'))\,dt',$$

and subtracting this from the identity at t, then splitting the integral at m, gives (3.13). For uniqueness, suppose $u(a) = v(a)$. Symmetry and bilinearity give $K(\tau)(u,u) - K(\tau)(v,v) =$

$K(\tau)(u+v, u-v)$. Thus $u-v$ satisfies Lemma 3.4 with $f=(u+v)/2$ and $g=0$. Its unique solution is zero. □

Proof of Lemma 3.6. Translate time to set $a=0$. Let E_2 be the closure of the Schwartz functions in $\sup_x \|f\|_{L^2(x+B_1)}$ [46, p. 29]. Hölder's inequality gives $\sup_x \|f\|_{L^2(x+B_1)} \lesssim \|f\|_{L^6}$; approximation by C_c^∞ therefore embeds L^6 continuously in E_2. By (3.1), $u \in C([0,b]; E_2)$, hence $u \in L^2((0,m); E_2)$ for each $m<b$, as Dubois's theorem requires. Let φ be a smooth compactly supported field on $[0,m]\times\mathbb{R}^3$, divergence-free at each time. To differentiate $\int u(t)\cdot\varphi(t)$, split its increment into the change in u with the test fixed and the change in the test. Apply (3.14) to the first part and continuity of u in L^6 to the second. This gives the continuous derivative

$$\int_{\mathbb{R}^3} \Big[u\cdot(\partial_t\varphi + \Delta\varphi) + (u\otimes u) : \nabla\varphi \Big]\, dx.$$

Integration from 0 to m gives Dubois's formulation (NSW) at time m, with datum $u(0)$ [46, p. 29], so u solves (NSW) on $(0,b)$. Theorem 1 of [46, pp. 29–30] then supplies a tempered distribution v_0 such that

$$u(t) = e^{t\Delta}v_0 - \int_0^t e^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}\big(u(t')\otimes u(t')\big)\,dt' \qquad (0\le t<b)$$

in $\mathcal{S}'(\mathbb{R}^3)$. Its continuous trace at $t=0$ gives $v_0=u(0)$. Since $u(t')\in L^6$, $\nabla u(t')\in L^2$ and $\operatorname{div} u=0$, Hölder's inequality gives $\operatorname{div}(u\otimes u)=(u\cdot\nabla)u\in L^{3/2}$. By (2.2)–(2.3), the integrand after the minus sign is $-K(t-t')(u(t'),u(t'))$. Lemma 3.3 gives its integral in X; testing against Schwartz fields, which act continuously on X by (3.1), identifies it with the distributional integral. This proves (2.4) for $t<b$. Continuity of both sides in X, from Subsection 3.1 and Lemma 3.3, includes $t=b$. □

Proof of Lemma 3.7. Let $L>0$ and $\tau>0$, let $f\in L^{3/2}(\mathbb{T}^3_L)$ have mean zero, and let $z = e^{\tau\Delta}\mathbb{P}f$. Since $|\hat z(k)| \le e^{-4\pi^2\tau|k|^2}L^{-3}\|f\|_{L^1(Q_L)}$, the field z is smooth; it is real and divergence-free, and it has mean zero. Thus z belongs to $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$, and so does the field $\Psi=-\Delta e^{\tau\Delta}z$, whose Fourier coefficients are $4\pi^2|k|^2e^{-4\pi^2\tau|k|^2}\hat z(k)$. Since $\mathbb{P}_k$ is a real symmetric matrix and $\mathbb{P}_k\hat z(k)=\hat z(k)$, Parseval's identity and (3.15) give

$$\int_{Q_L}\Psi\cdot f\,dx = L^3\sum_k 4\pi^2|k|^2e^{-4\pi^2\tau|k|^2}\hat z(k)\cdot\overline{\hat f(k)} = L^3\sum_k 4\pi^2|k|^2|\hat z(k)|^2 = \|z\|^2_{\dot H^1(\mathbb{T}^3_L)}.$$

Since $4\pi^2|k|^2\le\tau^{-1}e^{4\pi^2\tau|k|^2}$, the norms of Ψ in $L^2(Q_L)$ and in $\dot H^1(\mathbb{T}^3_L)$ are at most $\tau^{-1/2}$ and τ^{-1} times $\|z\|_{\dot H^1(\mathbb{T}^3_L)}$. By Hölder's inequality and (3.18) for Ψ,

$$\|z\|^2_{\dot H^1(\mathbb{T}^3_L)} \le \|f\|_{L^{3/2}(Q_L)}\|\Psi\|^{1/2}_{L^2(Q_L)}\|\Psi\|^{1/2}_{L^6(Q_L)} \lesssim \tau^{-3/4}\|f\|_{L^{3/2}(Q_L)}\|z\|_{\dot H^1(\mathbb{T}^3_L)},$$

which gives (3.19), immediately if $z=0$ and otherwise by division by $\|z\|_{\dot H^1}$. The identity $e^{t\Delta}e^{\tau\Delta}\mathbb{P}f = e^{(t+\tau)\Delta}\mathbb{P}f$ holds coefficient by coefficient. For continuity at $\tau_0>0$, write the field near τ_0 as $e^{(\tau-\tau_0/2)\Delta}(e^{(\tau_0/2)\Delta}\mathbb{P}f)$. The inner field lies in $\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$, so strong continuity of the heat flow applies from both sides of τ_0.

Let $u,v\in\dot H^1_{\mathrm{div}}(\mathbb{T}^3_L)$. By Hölder's inequality and (3.18),

$$\|(u\cdot\nabla)v\|_{L^{3/2}(Q_L)} \le \|u\|_{L^6(Q_L)}\|\nabla v\|_{L^2(Q_L)} \lesssim \|u\|_{\dot H^1(\mathbb{T}^3_L)}\|v\|_{\dot H^1(\mathbb{T}^3_L)},$$

and the same holds with u and v exchanged. Because u is divergence-free, $(u\cdot\nabla)v=\operatorname{div}(v\otimes u)$. Its integral over a period cell vanishes by testing this periodic distributional divergence against the constant function; the same holds with u,v exchanged. The first part of the lemma, for $f=(u\cdot\nabla)v+(v\cdot\nabla)u$, therefore gives (3.20). □

A.2. Proof of Lemma 2.3. Tao's unforced H^1 theory takes solenoidal H^1 data and gives a local mild solution in $L^\infty_t H^1_x \cap L^2_t H^2_x$ with pressure determined by the velocity [84, published pp. 29, 31, (9), (11)]. Corollaries 5.2 and 5.8 give the continuation alternatives.

The given solution is continuous only in the homogeneous gradient norm. In the whole space the energy identity supplies the missing L^2 bound. On the torus Poincaré's inequality supplies it from the zero mean. Uniqueness identifies the H^1 solution in each case.

Proof of Lemma 2.3. Let u satisfy the first assertion, set $u_0 = u(0)$ and fix $0 < b < T$. Obtain a local H^1 mild solution $\tilde{u}$ from u_0 on $[0, b']$, $0 < b' \le b$, by [84, Theorem 5.4(ii)]. By [84, Theorem 5.4(i)], $\tilde{u}$ is continuous in H^1 and hence in X by solenoidality and $H^1 \subset L^6$. Lemma 3.3 integrates $K(t - t')(\tilde{u}(t'), \tilde{u}(t'))$ from [84, (11)] in X, continuously in t. Schwartz tests identify the distributional integral, and continuity in X extends the identity to every time. Lemma 3.5 gives $\tilde{u} = u$ on $[0, b']$. Since $u_0 \in C^\infty_{c,\mathrm{div}}$ is Schwartz, $\tilde{u}, \tilde{p}$ are smooth on $[0, b'] \times \mathbb{R}^3$ with every time and space derivative bounded in L^2 [84, Theorem 5.4(iv)]. Equation (10) of [84] gives (1.1). Testing with $\tilde{u}$ and integrating by parts gives $\frac{d}{dt}\|\tilde{u}(t)\|^2_{L^2} = -2\|\nabla\tilde{u}(t)\|^2_{L^2}$, since solenoidality cancels pressure and transport. Thus $\|\tilde{u}(t)\|_{L^2} \le \|u_0\|_{L^2}$ and

$$\|\tilde{u}(t)\|_{H^1} \le \|u_0\|_{L^2} + \max_{[0,b]} \|u\|_{\dot{H}^1} \qquad (0 \le t \le b').$$

Continuity of u into X on $[0, b]$ bounds the right side independently of b'. If the maximal H^1 time were $T_* \le b$, Lemma 3.5 would give $\tilde{u} = u$ on each $[0, b']$ with $b' < T_*$. The displayed bound would contradict the blow-up alternative [84, Corollary 5.8, published pp. 56–57]. The smooth solution thus agrees with u on $[0, b]$ and has L^2 norm at most $\|u_0\|_{L^2}$, for arbitrary $b < T$.

For smooth mean-zero solenoidal u_0 on $\mathbb{T}^3$, apply the periodic H^1 theory [84, Theorem 5.1(i),(ii),(iv)] with $K_{\mathbb{T}}$ replacing K. Continuity in H^1 and preservation of the mean [84, (39)] give continuity in $\dot{H}^1_{\mathrm{div}}(\mathbb{T}^3)$. Lemma 3.7 permits the use of Lemmas 3.3 and 3.5, identifying this velocity with u. If the alternative [84, Corollary 5.2] gave blow-up at $T_* \le b < T$, (3.16) would give $\|\tilde{u}(t)\|_{H^1} \lesssim \max_{[0,b]} \|u\|_{\dot{H}^1}$ for $t < T_*$, excluding blow-up. The solution is therefore smooth on $[0, b]$ by Theorem 5.1(iv), with periodic pressure by the periodic H^1 definition [84, p. 29]. This holds for every $b < T$. □

Appendix B. From represented data to the analytic certificate

To apply (7.18) one must bound *every* admissible periodic perturbation by its forcing and constraints. Only the input fields have finite descriptions. Even a single time and angle harmonic of the perturbation is still a function of radius and height. The cell inequalities retain polynomial moments of such functions and bound the discarded degrees, the other harmonics and the spatial exterior. In (B.20)–(B.21) their combined lower bound cancels the density integral and the forcing-field pairings in the multiplier estimate.

In Proposition B.14 the conditions on the decoded data are given by `CellSystemCondition` at revision `ad53419`. These specify, among other things, the coefficient models and their remainders, the joins across $y_3 = 0$, the pressure freedoms and the common coordinates for neighbouring cells. The certificate supplies a coercive estimate at the reference field, a bound for transferring it to the approximate field, a time-slice bound and two adjoint estimates for the parameter update.

Estimate at the reference field, (7.18). The cell and tree conditions and the bounds on rows, tails and the exterior yield (B.21) in Proposition B.8. The multiplier enclosures bound the remaining pairing in (B.20). Lemma B.2 then gives (7.18) with the constants in (B.22).

Change of field and time-slice energy, (7.19) and (7.22). The bounds on the change of field and the reference quadratic-form bound enter Lemma B.9. The first estimate transfers (7.18) to W^{a}_{α} in (7.52). The second gives a gradient bound at each phase in (7.55).

Removing the forces, (7.32)–(7.37). Lemmas B.10–B.12 bound the quadratic and linear pairings with ψ_j. Lemma B.13 uses the interior expansion and exterior residual bound for the prescribed part of the jth parameter update. The remaining scalar and integral conditions include the residual budget, nonvanishing and the parameter inclusions. Their values must come from the particular data for which the theorem is invoked.

B.1. The form of the data of a certificate. The approximate, reference, forcing and adjoint fields are finite harmonic sums in normalized time θ and angle about the y_3 axis. Their coefficients combine piecewise polynomials with the radial phase at $|y|^2$ in (B.5). For the choice in Subsection C.1 this phase carries the logarithmic oscillations in the far field. The constraints Ξ_i and fields e_k use cylindrical polynomials on finitely many meridian cells. The weight of (7.14) is constant: $w \equiv w_* = \frac{39}{50}$.

For the symmetries, use cylindrical coordinates $y = (y_1, y_2, y_3)$, $r = (y_1^2 + y_2^2)^{1/2}$, $z = y_3$ and $y_1 + iy_2 = re^{i\phi}$, with frame $\mathbf{e}_r = (\cos\phi, \sin\phi, 0)$, $\mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0)$ and $\mathbf{e}_z = (0, 0, 1)$. For an angle ϕ_0 we write R_{ϕ_0} for the rotation by ϕ_0 about the y_3 axis, and we write R_* for the rotation by π about the y_1 axis, $R_* y = (y_1, -y_2, -y_3)$. Let $k_{\mathrm{rot}} \geq 3$ be the integer angular order. The authors write $\varkappa$ for this integer. We distinguish it from the decay exponent $0 < \kappa < 1/4$ in Sections 2–6. The set Γ contains the two pairs

$$(R_{\pi/k_{\mathrm{rot}}}, \tfrac{1}{2}) \qquad \text{and} \qquad (R_*, 0), \tag{B.1}$$

and each of its elements is of the form (R_{ϕ_0}, θ_0) or $(R_{\phi_0} R_*, \theta_0)$, where the angle $\phi_0 \geq 0$ is an integer multiple of π/k_{rot} and $\theta_0 - \frac{k_{\mathrm{rot}}\phi_0}{2\pi} \in \mathbb{Z}$. Hence $m = 2$ in (7.6). The harmonics are indexed by $\iota = (\iota_1, \iota_2) \in \mathbb{Z}^2$:

$$E_\iota(\theta, \phi) = \exp\big(i\,[\,2\pi(2\iota_2 - \iota_1)\,\theta + k_{\mathrm{rot}}\,\iota_1\,\phi\,]\big). \tag{B.2}$$

These are the functions $e^{i(2\pi j\theta + j'\phi)}$ with integers j and j' such that k_{rot} divides j' and $j - j'/k_{\mathrm{rot}}$ is even, that is, those which do not change when θ increases by $\frac{1}{2}$ and ϕ by π/k_{rot}. Indeed, $j = 2\iota_2 - \iota_1$ and $j' = k_{\mathrm{rot}}\iota_1$; conversely, take $\iota_1 = j'/k_{\mathrm{rot}}$ and $\iota_2 = (j + \iota_1)/2$. Thus a field whose components in the frame $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$ are combinations of the functions E_ι with coefficients that depend only on (r, z) satisfies (7.5) for the first pair of (B.1). We let $\iota^\flat = (-\iota_1, \iota_2 - \iota_1)$, so that $E_{\iota^\flat}(\theta, \phi) = E_\iota(\theta, -\phi)$.

The fields of the first kind are written in the variables

$$\sigma = \frac{|y|}{|y| + \mathsf{L}} \in [0, 1), \qquad \eta = \frac{y_3}{|y|} \in [-1, 1] \tag{B.3}$$

and ϕ, where $\mathsf{L} > 0$ is a length, and with the unit vector $y/|y|$ and the vectors

$$\mathbf{p} = \Big(\frac{\eta\, y_1}{|y|}, \frac{\eta\, y_2}{|y|}, -(1 - \eta^2)\Big), \qquad \mathbf{a} = \Big(-\frac{y_2}{|y|}, \frac{y_1}{|y|}, 0\Big), \tag{B.4}$$

which are the unit vectors of the spherical coordinates in the directions of the polar angle and of ϕ, multiplied by $(1 - \eta^2)^{1/2}$. On the axis set $\phi = 0$. At $y = 0$ use $\eta = 0$, $y/|y| = 0$, $\mathbf{p} = -\mathbf{e}_z$ and $\mathbf{a} = 0$. For $\mathsf{L} > 0$, $|y| = \mathsf{L}\sigma/(1 - \sigma)$. The variable σ compactifies the radius, and $|y|\,\partial_{|y|} = \sigma(1 - \sigma)\,\partial_\sigma$. Let f be a real function of class C^3 on a neighborhood of $[0, \infty)$ such

that $\rho f'(\rho)$, $\rho^2 f''(\rho)$ and $\rho^3 f'''(\rho)$ are bounded for $\rho \geq 0$, for instance a multiple of $\log(1+\rho)$; we call it the radial phase. For $\iota \in \mathbb{Z}^2$, an integer l and a function u of (σ, η) let

$$E_{\iota,l}(\theta, y) = E_\iota(\theta, \phi)\, e^{il f(|y|^2)}, \qquad \mathcal{V}_l \mathsf{u} = \sigma(1-\sigma)\, \partial_\sigma \mathsf{u} + 2i\, l\, |y|^2 f'(|y|^2)\, \mathsf{u}, \tag{B.5}$$

so that $|y|\, \partial_{|y|}(e^{il f(|y|^2)} \mathsf{u}) = e^{il f(|y|^2)} \mathcal{V}_l \mathsf{u}$. A field of the first kind is a field of the form

$$\operatorname{Re} \sum_{(\iota,l) \in \mathcal{A}} E_{\iota,l}(\theta, y) \Big[\mathsf{u}_0 \frac{y}{|y|} + \mathsf{u}_1\, \mathbf{p} + \mathsf{u}_2\, \mathbf{a} \Big], \tag{B.6}$$

where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^2 \times \mathbb{Z}$ and $\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2$ are functions of (σ, η) which depend on (ι, l) and are formed as follows from two piecewise polynomials, in such a way that every term of the sum is divergence-free. If $\iota_1 \neq 0$, then, with integers $h_0 \geq 2$ and $h \geq 1$ and piecewise polynomials A and B,

$$\begin{gathered} \mathsf{u}_0 = \sigma^{h_0}(1-\sigma)^h (1-\eta^2)^2 \mathsf{A}, \qquad \mathsf{u}_1 = \sigma^{h_0}(1-\sigma)^h (1-\eta^2)^2 \mathsf{B}, \\ \mathsf{u}_2 = \frac{i}{k_{\mathrm{rot}}\, \iota_1} \Big[(\mathcal{V}_l + 2)\, \mathsf{u}_0 - \partial_\eta \big((1-\eta^2)\, \mathsf{u}_1 \big) \Big]. \end{gathered} \tag{B.7}$$

If $\iota_1 = 0$, then, with an integer $h \geq 1$, a number $\gamma \in \mathbb{C}$, piecewise polynomials S and T, and with $\Psi = \gamma\, \mathsf{g}(\sigma)\, \eta + \mathsf{S}$,

$$\mathsf{u}_0 = -\sigma(1-\sigma)^h\, \partial_\eta \big((1-\eta^2)\, \Psi \big), \quad \mathsf{u}_1 = -(\mathcal{V}_l + 2)\big(\sigma(1-\sigma)^h\, \Psi \big), \quad \mathsf{u}_2 = \sigma(1-\sigma)^h\, \mathsf{T}, \tag{B.8}$$

so that Ψ plays the part of a stream function and T that of a swirl. Here g is a real function on $[0,1]$ with $\mathsf{g}(0) = 1$ which is a polynomial on an interval $[0, \sigma_{\mathsf{g}}]$ and constant on $[\sigma_{\mathsf{g}}, 1]$, of class C^2. In both cases the divergence of the term (ι, l) of (B.6) is $|y|^{-1} E_{\iota,l}$ times $(\mathcal{V}_l + 2)\, \mathsf{u}_0 - \partial_\eta((1-\eta^2)\, \mathsf{u}_1) + i\, k_{\mathrm{rot}}\, \iota_1\, \mathsf{u}_2$, which vanishes identically. In (B.7), the last term cancels the first two since $i^2 = -1$. In (B.8), cancellation follows because $\mathcal{V}_l$ commutes with ∂_η and with multiplication by functions of η. We call the set $\mathcal{A}$, the function g and, for each $(\iota, l) \in \mathcal{A}$, the exponents, the number γ and the piecewise polynomials the *data* of the field (B.6); with the length L and the radial phase f they determine it.

For each amplitude, the σ and η partitions divide $[0,1] \times [-1,1]$ into rectangles. On each rectangle the amplitude is a complex polynomial. Across a common edge the two polynomial traces agree, as do their derivatives normal to that edge up to the prescribed order. The orders in (σ, η) are $(2,1)$ for A, $(1,2)$ for B, $(2,2)$ for S and $(1,1)$ for T. These matches keep all components of (B.7) and (B.8) continuously differentiable when a derivative in the formulas crosses a partition edge. In addition, S and T vanish at $\sigma = 0$. The field (B.6) is then of class C^1 on $\mathbb{R} \times \mathbb{R}^3$ and has period 1 in θ; its second derivatives are bounded functions, which are the classical derivatives away from the spheres and the cones about the y_3 axis that correspond to the points of the partitions and away from the sphere $\sigma = \sigma_{\mathsf{g}}$. Continuity of the first derivatives across the interfaces makes this piecewise Hessian the weak Hessian. These are the derivatives used to check the approximate equation. Proposition 7.11 obtains smoothness of the corrected field from the exact equation. The set $\mathcal{A}$ contains $(-\iota, -l)$ with (ι, l), with the complex conjugate data, so that the terms of (B.6) come in conjugate pairs; and it contains $(\iota^\flat, l)$ with (ι, l), with $\mathsf{A}(\sigma, -\eta)$ and $-\mathsf{B}(\sigma, -\eta)$ in place of $\mathsf{A}(\sigma, \eta)$ and $\mathsf{B}(\sigma, \eta)$, while S and T are odd in η. With these relations the field is invariant under the second pair of (B.1), and it is invariant under the first because its harmonics are those of (B.2). It is therefore Γ-invariant, since every element of Γ is composed of powers of these two pairs and of a translation of θ by an integer.

The behaviour at the axis, at the origin and at infinity is prescribed by explicit factors. For $\iota_1 \neq 0$ the factor $(1-\eta^2)^2 = (r/|y|)^4$ makes each component vanish on the axis, and σ^{h_0} with

$h_0 \geq 2$ makes it vanish to second order at the origin. For $\iota_1 = 0$ the vectors $\mathbf{p}$ and $\mathbf{a}$ of (B.4) vanish on the axis, and at the origin only the term with γ contributes to the gradient, by the linear field $-(\gamma/\mathsf{L})\, E_{\iota,l}(\theta,0)\,(y_1, y_2, -2y_3)$. The factor $(1-\sigma)^h = (\mathsf{L}/(|y|+\mathsf{L}))^h$ gives the decay: with the conditions on f, if $h \geq h_*$ in every term, then the field (B.6) and its derivative in θ are bounded by a constant multiple of $(1+|y|)^{-h_*}$, and its first and second derivatives in y by constant multiples of $(1+|y|)^{-h_*-1}$ and $(1+|y|)^{-h_*-2}$. For $h_* = 1$ these contain the bounds (7.8).

The approximate profile depends on the parameter linearly through the coefficients and, in addition, through the radial phase. Let $\mathsf{k} \in \mathbb{R}^n$, and let χ be a piecewise polynomial function on $[0,1]$ of class C^2 such that $\chi = 0$ on $[0, \sigma_\chi]$, for a point $\sigma_\chi \in (0,1)$ of its partition, and $\chi = 1$ near 1. Then

$$W^{\mathrm{a}}_\alpha = W^0_\alpha + \sum_{i=1}^n \alpha_i\, W^i_\alpha, \qquad f_\alpha = f + (\mathsf{k}\cdot\alpha)\, f_*, \qquad f_*(|y|^2) = \int_0^{|y|} \chi\Big(\frac{\rho}{\rho+\mathsf{L}}\Big)\frac{d\rho}{\rho}, \tag{B.9}$$

where $W^0_\alpha, \dots, W^n_\alpha$ are fields of the form (B.6) with one length L, with the radial phase f_α in place of f, and with data that do not depend on α; each of them has its own set $\mathcal{A}$, partitions and function g. The cutoff gives $f_* = 0$ for $|y| \leq r_\chi = \mathsf{L}\sigma_\chi/(1-\sigma_\chi)$, and integration where $\chi = 1$ gives $f_*(|y|^2) = \log|y| + \text{constant}$ for large $|y|$. Hence the fields in (B.9) are fixed in the ball and W^{a}_α depends affinely on α there. For $|y| > 0$, differentiation gives $2|y|^2 f_*'(|y|^2) = \chi(\sigma)$. The phase change therefore adds $l\,(\mathsf{k}\cdot\alpha) f_*(|y|^2)$ to each phase and $i\,l\,(\mathsf{k}\cdot\alpha)\chi(\sigma)$ to $\mathcal{V}_l$. The field is C^1 in (α,θ,y). The bounds (7.8) are uniform on $\mathcal{K}$: its coordinates are bounded by (7.29), using $d_j \neq 0$ for $j \notin \mathcal{J}$. The period and the coefficients of the force are given by $1/S(\alpha) = 1/S(0) + \varsigma\cdot\alpha$ and $\beta^{\mathrm{a}}(\alpha) = \beta^0 + \sum_i \alpha_i\,\beta^i$, with $\varsigma \in \mathbb{R}^n$ and $\beta^0, \dots, \beta^n \in \mathbb{R}^n$. The two linear forms $\mathsf{k}\cdot\alpha$ and $\varsigma\cdot\alpha$ control the far-field phase and the reciprocal period. If the leading tail oscillates with $\log|y|$, varying the period changes its time derivative. For the nonzero mode in Subsection C.4, an unmatched radial phase leaves a transport term of order $|y|^{-1}$, which is not square integrable. The calculation there gives the required relation between the two slopes.

To define the residual F^{a}_α, compute the left-hand side of (7.10) with the second derivatives above and add $\nabla(q_\alpha - \frac{1}{2}|W^{\mathrm{a}}_\alpha|^2)$, where $q_\alpha = q^0 + \sum_i \alpha_i\, q^i$. The auxiliary total pressure is q_α, so the pressure in this representative of the residual is $q_\alpha - \frac{1}{2}|W^{\mathrm{a}}_\alpha|^2$. Each q^i is the real part of a finite sum of terms $E_{\iota,l}\,\sigma^{h_0}(1-\sigma)^h(1-\eta^2)^{h'}\mathsf{P}$, with the functions $E_{\iota,l}$ of (B.5) for the phase f_α and with continuous piecewise polynomials P. The exponents are integers $h_0 \geq 0$ and $h \geq 2$, and $h' \geq 1$ if $\iota_1 \neq 0$, $h' = 0$ if $\iota_1 = 0$. If $h_0 = 0$, then at $\sigma = 0$ the function P vanishes if $\iota_1 \neq 0$ and is constant if $\iota_1 = 0$. The terms come with those of $(-\iota, -l)$, with the complex conjugate data, and of $(\iota^\flat, l)$, with $\mathsf{P}(\sigma, -\eta)$ in place of $\mathsf{P}(\sigma,\eta)$. These exponent and matching conditions give continuity and spatial Lipschitz bounds for q^i on each ball, uniformly in θ. Their gradients are therefore locally bounded and pair to zero with compactly supported divergence-free tests, by integration by parts. Together with the bounded weak second derivatives above, this gives a locally bounded residual satisfying (7.10).

The reference profile does not depend on α: $W^{\mathrm{r}}_\alpha = W^{\mathrm{r}}$ for every $\alpha \in \mathcal{K}$, where W^{r} is one field of the form (B.6), with a length, a radial phase and data of its own and with the integer k_{rot} of (B.1). Its piecewise polynomials are given in a basis of Hermite type: in each variable and on each interval of the partition, by their derivatives up to an order $D-1$ at the two ends and by finitely many further coefficients, where the integer D exceeds by at least one the order to which they are joined. The forcing fields are also of the first kind: b_i is a field of the form (B.6), with a length L_i and a radial phase of its own, with $\gamma = 0$ and with piecewise polynomials that vanish for $\sigma \geq \sigma_i$, for a point $\sigma_i < 1$ of their partitions. Thus b_i is of class C^1, divergence-free and Γ-invariant, and it vanishes for $|y| \geq \mathsf{L}_i\sigma_i/(1-\sigma_i)$. The radius is obtained

from $\sigma = |y|/(|y| + \mathsf{L}_i)$; continuity includes the boundary sphere. The adjoint fields do not depend on α either: for $j \in \mathcal{J}$ and every $\alpha \in \mathcal{K}$ the field $\psi_{\alpha,j}$ is one field ψ_j of the form (B.6), with one length, one radial phase and one function g for all j and with $h \geq 3$ in every term. Nor do the rows of the matrix $Y(\alpha)$ with index in $\mathcal{J}$, which alone enter the conditions of a certificate, depend on α; we write Y_{jl} for their entries.

The fields of the constraints and the fields e_k are given on cells. A cell is a rectangle $\mathsf{Q} = [r_1, r_2] \times [z_1, z_2]$ with $0 \leq r_1 < r_2$ and $0 \leq z_1 < z_2$ in the variables (r, z); it stands for the set of the points y with $(r, |y_3|) \in \mathsf{Q}$, which contains its mirror image in the plane $y_3 = 0$. We are given finitely many cells with disjoint interiors, which cover a square $[0, r_*]^2$ such that every forcing field vanishes for $|y| \geq r_*$. On each cell Q, and for each ι in a finite set which contains $-\iota$ and $\iota^\flat$ with ι, the field Ξ_i is given by polynomials $\Xi_{i,ab}^{\mathsf{Q},\iota}$ in (r, z), where $a, b \in \{r, \phi, z\}$, with $\Xi_{i,ab}^{\mathsf{Q},-\iota} = \overline{\Xi_{i,ab}^{\mathsf{Q},\iota}}$: for a field v as in (7.11),

$$\ell_i(v) = \int_0^1 \int_{\mathbb{R}^3} \sum_{\mathsf{Q}} \mathbf{1}_{\mathsf{Q}}(r, |y_3|) \sum_{\iota} \sum_{a,b} \mathrm{Re}\big[E_\iota(\theta, \phi)\, \Xi_{i,ab}^{\mathsf{Q},\iota}(r, z)\big]\, \mathbf{e}_b \cdot (\mathbf{e}_a \cdot \nabla) v \, dy \, d\theta, \tag{B.10}$$

where, for $z < 0$, the polynomial is replaced by $\Xi_{i,ab}^{\mathsf{Q},\iota^\flat}(r, -z)$, with the opposite sign if exactly one of the indices a and b is r. In (B.10), a indexes the differentiation direction and b the component of v. Thus the entry with paper indices (a, b) corresponds to entry (b, a) of the formal matrix `CellMatField.frameMap`, which takes the output index first. The finite polynomial representation on the cells gives a bounded Ξ_i that vanishes outside them. The prescribed value of the constraints on the perturbation is $\bar{\ell} \equiv 0$, so that the field v of Theorem 7.1 satisfies $\ell(v) = 0$. When $\mathcal{J}$ is not empty, the fields Ξ_i come with compactly supported vector fields $\Phi_1, \dots, \Phi_n$, which belong to $\mathcal{H}$ with their gradients and whose components in the frame $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$ are finite sums of harmonics of the form (B.2), with an integer of their own in place of k_{rot}, times r times piecewise polynomials in (r, z) of the Hermite type above, and with a number $c_\Xi \geq 0$ such that $\| \sum_i x_i \, (\Xi_i - \nabla \Phi_i) \|_{\mathcal{H}} \leq c_\Xi \, |x|$ for every $x \in \mathbb{R}^n$. Up to this error, which enters the numbers a_j, the constraint $\ell_i(v)$ is therefore the pairing $\langle \nabla v, \nabla \Phi_i \rangle_{\mathcal{H}}$ of v with a fixed compactly supported field.

The fields e_k are obtained by orthogonalization from fields with one harmonic in one cell. For each cell Q and each ι in a finite set, we are given finitely many triples $\varphi = (\varphi_r, \varphi_\phi, \varphi_z)$ of polynomials in (r, z), with real coefficients if $\iota = 0$, which satisfy $r\, \partial_r \varphi_r + \varphi_r + i\, k_{\mathrm{rot}}\, \iota_1\, \varphi_\phi + r\, \partial_z \varphi_z = 0$, the equation of a divergence-free field with the harmonic E_ι, and $\varphi_r = \varphi_\phi = 0$ on the axis if the cell meets it. For each cell and each ι the Hermitian matrix of the integrals $\int_{\mathsf{Q}} \varphi \cdot \overline{\varphi'}\, r \, dr \, dz$, where φ and φ' run through the triples, is positive definite. Each triple gives the fields which vanish at the points with $(r, |y_3|) \notin \mathsf{Q}$ and are equal, at the other points, to the real part and, if $\iota \neq 0$, to the imaginary part of $E_\iota \, (\varphi_r \, \mathbf{e}_r + \varphi_\phi \, \mathbf{e}_\phi + \varphi_z \, \mathbf{e}_z)$ where $z \geq 0$, and of $E_{\iota^\flat} \, (\varphi_r \, \mathbf{e}_r - \varphi_\phi \, \mathbf{e}_\phi - \varphi_z \, \mathbf{e}_z)$, with the polynomials taken at the point $(r, -z)$, where $z < 0$. Let $\tilde{e}_1, \dots, \tilde{e}_N$ be these fields, for one ι of each pair $\pm\iota$, and let G be their Gram matrix in $\mathcal{H}$, which is then positive definite. Then

$$e_k = \tau_k^{1/2} \sum_{k'=1}^{N} \big(G^{-1/2}\big)_{kk'} \, \tilde{e}_{k'}, \tag{B.11}$$

where $\tau_k > 0$ depends only on the cell and on the harmonic of $\tilde{e}_k$, and $\frac{3}{50}\, \tau_k \leq 400\, (1 + 2^{-18})$. The fields e_k are pairwise orthogonal, with $\|e_k\|_{\mathcal{H}}^2 = \tau_k$. Indeed, $G^{-1/2} G G^{-1/2} = I$.

A cellwise solenoidal polynomial can have nonzero normal trace on a side. Its zero extension can therefore have a distributional divergence there. The pairings in $[F]$ use the representative F in (7.18); replacing it by $\mathbb{P}F$ would require further justification.

B.2. The multiplier inequality. To bound a perturbation by its forcing and constraints in (7.18), Proposition B.1 pairs the linearized equation with a multiplier built from the perturbation. Its lower bound contains a quadratic density that can have a negative part. Subsection B.3 gives the lower bound (B.21) for its integral. Adding that bound to (B.20) cancels both the integral and the pairings with the forcing fields. The remaining terms give the norm of (v, β) (Lemma B.2).

The two patterns in Subsection C.1 turn in opposite directions. A mode nearly at rest in one frame can have high frequency in the other. To use this in the estimate, (B.14) assigns a sign to the skew part of each multiplier according to the modes on which it acts. For a field v and $\iota \in \mathbb{Z}^2$ let

$$\hat{v}_\iota(\theta, y) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \overline{E_\iota(\theta', \phi')}\, R_{-\phi'}\, v(\theta + \theta', R_{\phi'} y)\, d\phi'\, d\theta' \tag{B.12}$$

be the mode of v with the harmonic E_ι, in the frame that turns with the angle; in a Γ-invariant field only these modes occur. A change of variables in (B.12) gives, for all θ_0 and ϕ_0,

$$\hat{v}_\iota(\theta + \theta_0, R_{\phi_0} y) = E_\iota(\theta_0, \phi_0)\, R_{\phi_0}\, \hat{v}_\iota(\theta, y), \tag{B.13}$$

so that the Euclidean norms of $\hat{v}_\iota$, of its curl and of their derivatives in y depend only on r and y_3.

Partition the harmonics (B.2) into finitely many classes. Each class carries functions D_ι and Z_ι on $\mathbb{R}^3$, with the same functions for $-\iota$. There are two finite index sets $\mathcal{P}_\lambda$ and $\mathcal{P}_\mu$. Each index p carries a harmonic set $\mathcal{N}_p$ disjoint from $-\mathcal{N}_p$, a rotation rate $\nu_p(\alpha)$, a nonnegative weight λ_p or μ_p, a transport bound for $|W^{\mathrm{r}} + \frac{1}{2} y - \nu_p \mathbf{e}_z \times y|$, and positive auxiliary functions. A μ index also carries a positive auxiliary number; there is a common $c_\mu \in (0, 1)$. All these functions are axisymmetric. They are given in the coordinates (B.3), with their own length and separate expressions on $y_3 \geq 0$ and $y_3 < 0$. The functions $\mathsf{D}_\iota, \mathsf{Z}_\iota$ are piecewise polynomials in $(\sigma, |\eta|)$ of class C^2, constant near the origin, with $\mathsf{D}_\iota \geq 0$. The weight λ_p is $1 - \sigma$ times a continuous piecewise polynomial constant at $\sigma = 0$; the weight μ_p has the same form with $(1 - \sigma)^2$. The auxiliary functions are integer powers of $1 - \sigma$ times functions constant on partition rectangles. Only $\nu_p(\alpha)$ depends on α. Let j_* be an integer such that every forcing harmonic satisfies both $|\iota_2| \leq j_*$ and $|\iota_2 - \iota_1| \leq j_*$. No $\mathcal{N}_p$ contains a harmonic in this range. Use the same σ partition on both sides of $y_3 = 0$. At $\eta = 0$ the one-sided $|\eta|$ derivatives of $\mathsf{D}_\iota, \mathsf{Z}_\iota$ agree at orders zero and two and are opposite at order one. The two values agree for each weight λ_p, μ_p. Require $\mathsf{D}_\iota, \mathsf{Z}_\iota \in C^2(\mathbb{R}^3)$ and globally Lipschitz λ_p, μ_p.

For a single mode z with derivative $i\Omega z$, ordinary energy testing gives $\mathrm{Re}((i\Omega z)\bar{z}) = 0$. Testing against $i\,\mathrm{sgn}(\Omega) z$ gives $\mathrm{Re}((i\Omega z)\overline{i\,\mathrm{sgn}(\Omega) z}) = |\Omega|\,|z|^2$. The two skew terms in (B.14) use this gain for the curl and velocity in rotating frames. The admissibility conditions choose the sign of the frequency on each selected set.

Let $\mathrm{sgn}_p(\iota) = 1$ if $\iota \in \mathcal{N}_p$, $\mathrm{sgn}_p(\iota) = -1$ if $-\iota \in \mathcal{N}_p$, and $\mathrm{sgn}_p(\iota) = 0$ otherwise. For $v \in \mathcal{D}^\Gamma_{S(\alpha)}$ the multiplier is the field $\mathsf{M}v \in \mathcal{H}_{\mathrm{div}}$ with

$$\begin{aligned} \langle \mathsf{M}v, v' \rangle_{\mathcal{H}} = \langle -\Delta v, v' \rangle_{\mathcal{H}} + \sum_\iota \int_0^1 \int_{\mathbb{R}^3} \mathrm{Re}\Big[& i \sum_{p \in \mathcal{P}_\lambda} \mathrm{sgn}_p(\iota)\, \lambda_p\, \mathrm{curl}\, \hat{v}_\iota \cdot \overline{\mathrm{curl}\, \hat{v}'_\iota} \\ & + i \sum_{p \in \mathcal{P}_\mu} \mathrm{sgn}_p(\iota)\, \mu_p\, \hat{v}_\iota \cdot \overline{\hat{v}'_\iota} \Big]\, dy\, d\theta \end{aligned} \tag{B.14}$$

for every smooth, divergence-free, 1-periodic test field v' with bounded spatial support. Thus $\mathsf{M}v$ is $-\Delta v$ plus, mode by mode, the divergence-free part of two skew terms: the curl term has order two and the velocity term order zero. There is at most one such field: the test fields of this kind, among them the products (7.3), span a dense subspace of $\mathcal{H}_{\mathrm{div}}$. Let

$$\Omega_p(\iota) = \frac{2\pi(2\iota_2 - \iota_1)}{S(\alpha)} + \nu_p(\alpha)\, k_{\mathrm{rot}}\, \iota_1 \tag{B.15}$$

be the frequency in the frame rotating at rate ν_p, and let

$$\|v\|_\mu^2 = \sum_{p\in\mathcal{P}_\mu} \sum_{\iota} \int_0^1 \int_{\mathbb{R}^3} |\operatorname{sgn}_p(\iota)|\, |\Omega_p(\iota)|\, \mu_p\, |\hat v_\iota|^2\, dy\, d\theta.$$

Proposition B.1 proves that this integral is finite. For the two frames of Subsection C.4, take $k_{\mathrm{rot}} = 19$ and $\nu_p = \pm 2\pi/(19\bar S)$. At the computed period $\bar S$ their frequencies in (B.15) are $4\pi\iota_2/\bar S$ and $4\pi(\iota_2 - \iota_1)/\bar S$. Both can have modulus at most $4\pi j_*/\bar S$ only when $|\iota_2| \le j_*$ and $|\iota_2 - \iota_1| \le j_*$. The two strips have different slopes, so their intersection contains finitely many integer pairs. Each such harmonic still has a coefficient depending on radius and height, the variables of the cells in Subsection B.3. For each actual period $S(\alpha)$ admissibility requires a positive lower bound for (B.15) on each selected set $\mathcal{N}_p$ (see (B.19) and the conditions preceding it).

A finite harmonic set contains 0, is closed under negation, and contains the spectrum of W^{r}. For each harmonic it carries bounds for the corresponding modes of W^{r}, ∇W^{r}, $\operatorname{curl} W^{\mathrm{r}}$ and $\nabla \operatorname{curl} W^{\mathrm{r}}$. Two further functions bound $|\operatorname{curl} W^{\mathrm{r}}|$ and $|\nabla \operatorname{curl} W^{\mathrm{r}}|_{\mathrm{op}}$. These bounds and the transport bounds are powers of σ and $1-\sigma$ times functions constant on rectangles in (B.3). For $\iota \in \mathbb{Z}^2$ set

$$\mathcal{I}_\iota = \mathsf{I}_{2,\iota}|\nabla^2 \hat v_\iota|^2 + \mathsf{I}'_{2,\iota}|\nabla \operatorname{curl} \hat v_\iota|^2 + \mathsf{I}'_{1,\iota}|\operatorname{curl} \hat v_\iota|^2 + \mathsf{I}_{1,\iota}|\nabla \hat v_\iota|^2 + \mathsf{I}_{0,\iota}|\hat v_\iota|^2, \tag{B.16}$$

The subscript is the derivative order; a prime marks a curl term. The five coefficients depend on y, the weights and these bounds. Before adding the p terms, they are $\mathsf{I}_{2,\iota} = \mathsf{D}_\iota - \frac{3}{50}$, $\mathsf{I}'_{2,\iota} = 1 - \mathsf{D}_\iota$, $\mathsf{I}'_{1,\iota} = \frac14 - \mathsf{Z}_\iota$, $\mathsf{I}_{1,\iota} = \mathsf{Z}_\iota - |\nabla^2 \mathsf{D}_\iota|_{\mathrm{op}} - \frac{3}{50} w$ and $\mathsf{I}_{0,\iota} = -|\nabla^2 \mathsf{Z}_\iota|_{\mathrm{op}}$, where $|\cdot|_{\mathrm{op}}$ is the matrix operator norm. Thus D_ι transfers part of the squared curl-gradient term to the Hessian term, and Z_ι transfers part of the squared curl term to the gradient term. The Hessians of D_ι and Z_ι bound the costs of these transfers.

For $p \in \mathcal{P}_\lambda$ with $\operatorname{sgn}_p(\iota) \neq 0$, the gain is $\lambda_p|\Omega_p(\iota)|$ in $\mathsf{I}'_{1,\iota}$; for $p \in \mathcal{P}_\mu$ it is $(1-c_\mu)\mu_p|\Omega_p(\iota)|$ in $\mathsf{I}_{0,\iota}$. The λ errors are subtracted from $\mathsf{I}'_{2,\iota}, \mathsf{I}'_{1,\iota}, \mathsf{I}_{1,\iota}, \mathsf{I}_{0,\iota}$, and the μ errors from $\mathsf{I}_{1,\iota}, \mathsf{I}_{0,\iota}$. They occur both on $\pm\mathcal{N}_p$ and at harmonics coupled to that set by a reference harmonic. The five coefficient formulas and their required lower bounds use the weight and its gradient, $|\nu_p|$, the transport and reference bounds, and the auxiliary functions and numbers. One subtraction can be seen directly. Fix a mode in $\pm\mathcal{N}_p$ for $p \in \mathcal{P}_\lambda$. At a point y, let b be the certificate bound on the relative transport speed and $d > 0$ its auxiliary coefficient for this product. Young's inequality bounds $(\lambda_p b + |\nabla\lambda_p|)|\nabla \operatorname{curl} \hat v_\iota|\,|\operatorname{curl} \hat v_\iota|$ by $d|\nabla \operatorname{curl} \hat v_\iota|^2 + (\lambda_p b + |\nabla\lambda_p|)^2 |\operatorname{curl} \hat v_\iota|^2/(4d)$. The first term uses part of $\mathsf{I}'_{2,\iota}$ wherever $\lambda_p + |\nabla\lambda_p| > 0$. The second is deducted from the frequency gain in $\mathsf{I}'_{1,\iota}$. For each parameter, the certificate bounds all five resulting coefficients from below on the regions assigned to each harmonic. One constant bounds from below the first four regional lower coefficients and (r^2+z^2) times the fifth. Resolved and eliminated harmonics have the corresponding bounds on each cell.

The five coefficients in (B.16). To the five baseline values above add the increments for the curl multipliers and the velocity multipliers. Both use the same sums over coupled harmonics. Fix y, α, ι and one multiplier p. Write $N = \mathcal{N}_p$, $\chi = \mathbf{1}_{\{\iota \in N \cup (-N)\}}$, $\omega_p = |\Omega_p(\iota)|$, b for the relative

transport bound and $\nu = \nu_p(\alpha)$. On the finite reference spectrum H, denote the bounds for the harmonics of $W^{\mathrm{r}}, \nabla W^{\mathrm{r}}, \operatorname{curl} W^{\mathrm{r}}, \nabla \operatorname{curl} W^{\mathrm{r}}$ by $U_\beta, G_\beta, C_\beta, \mathcal{D}_\beta$, respectively. Write $G_\Sigma = \sum_{\beta\in H} G_\beta$, and $C_*, \mathcal{D}_*$ for the two bounds on the full reference vorticity and its gradient. For any one of these arrays A, set

$$I_A = \sum_{\substack{\sigma=\pm 1\\ \sigma\iota\in N}} \sum_{\substack{\beta\in H\\ \sigma\iota-\beta\notin N}} A_\beta, \qquad O_A = \sum_{\substack{\sigma=\pm 1\\ \sigma\iota\notin N}} \sum_{\substack{\beta\in H\\ \sigma\iota+\beta\in N}} A_\beta.$$

In I_A the selected harmonic receives a coupling from outside N; in O_A the reference harmonic carries it into N. Both sums run over the two signs, including when $\iota = 0$. For either weight $a = \lambda_p$ or μ_p, abbreviate $Q_d(a) = (ab + |\nabla a|)^2/(4d)$ and $J(a) = \mathbf{1}_{\{a+|\nabla a|>0\}}$. The increments are pointwise, with derivatives of Lipschitz weights understood almost everywhere.

Curl multipliers. For a curl multiplier write $a = \lambda_p$ and denote its positive auxiliary functions by $d, e, L, s_0, t_0, t_1, t_2, t_3$ (the two shares, length, scale and four Young parameters). Define

$$B = |\nu| + \tfrac{1}{2}\big((1+t_0)C_* + L\mathcal{D}_* + t_1 I_G + t_2 I_{\mathcal{D}} + t_3 I_C\big).$$

The Hessian coefficient I_2 receives no increment. Suppressing the index ι, the four others receive

$$\begin{aligned}
\Delta\mathsf{I}_2' &= -d\chi J(a) - e\,\mathbf{1}_{\{a>0\}}\, O_U/s_0,\\
\Delta\mathsf{I}_1' &= \chi(a\omega_p - Q_d(a) - a^2 s_0 I_U/(4e) - aB) - aO_G/(2t_1),\\
\Delta\mathsf{I}_1 &= -\tfrac{a}{2}(\chi C_*/t_0 + O_C/t_3),\\
\Delta\mathsf{I}_0 &= -\tfrac{a}{2}(\chi\mathcal{D}_*/L + O_{\mathcal{D}}/t_2).
\end{aligned}$$

Velocity multipliers. For a velocity multiplier write $a = \mu_p$, with positive share d, Young parameter t_4 and number $\eta_Y > 0$. Its only increments are

$$\begin{aligned}
\Delta\mathsf{I}_1 &= -d\chi J(a) - aO_U/(2t_4),\\
\Delta\mathsf{I}_0 &= \chi((1-c_\mu)a\omega_p - Q_d(a) - a|\nu| - t_4 a I_U/2 - \eta_Y a^2/2)\\
&\quad - \chi G_\Sigma(G_\Sigma - I_G)/(2\eta_Y) - G_\Sigma O_G/(2\eta_Y).
\end{aligned}$$

For the first two curl increments, consider $i \in N$ coupled to $j = i - \beta \notin N$ by a reference mode β. Where $a > 0$, put $X = |\nabla \operatorname{curl} \hat{v}_j|$ and $Y = |\operatorname{curl} \hat{v}_i|$. Young's inequality gives $aU_\beta XY \le (e/s_0)U_\beta X^2 + (a^2 s_0/(4e))U_\beta Y^2$. The first cost goes to the source mode j, the second to the selected mode i. Summing gives the O_U loss in I_2' and the I_U loss in I_1', respectively. When $a = 0$ this coupling costs nothing. The pointwise multiplier formulas use the pair (time frequency, angular frequency), which is $(2\iota_2 - \iota_1, k_{\mathrm{rot}}\iota_1)$ in (B.2).

The data also obey a reflection symmetry. Under $(\iota, y) \mapsto (\iota^\flat, R_* y)$, permutations of the λ and μ indices carry $\mathcal{N}_p$ to $\{\iota^\flat : \iota \in \mathcal{N}_p\}$ and ν_p to $-\nu_p$. The functions $\mathsf{D}_\iota, \mathsf{Z}_\iota$, the weights, transport and reference bounds, and auxiliary data are unchanged. The sign is preserved. So is Ω_p: the time frequency $2\iota_2 - \iota_1$ in (B.2) is unchanged under $\iota \mapsto \iota^\flat$, while the angle frequency and the rotation rate both change sign. Substitution in the coefficient formulas gives

$$\text{the coefficients of } \mathcal{I}_{\iota^\flat} \text{ at } R_* y \text{ are those of } \mathcal{I}_\iota \text{ at } y. \tag{B.17}$$

The reflection identities for the piecewise coefficients hold off $y_3 = 0$, which suffices for (B.17) in the density integral.

The stretching term is

$$J = -\operatorname{curl} v \cdot \mathbb{S} \operatorname{curl} v + \big((v\cdot\nabla)\operatorname{curl} W^{\mathrm{r}}\big)\cdot \operatorname{curl} v + \frac{1}{2}\sum_{k=1}^{3} \nabla(\operatorname{curl} W^{\mathrm{r}})_k \cdot \big(v \times \partial_k v\big), \tag{B.18}$$

where $\mathbb{S}$ is the symmetric part of ∇W^{r} and $(\operatorname{curl} W^{\mathrm{r}})_k$ its kth vorticity component. Neither (B.16) nor J has a fixed sign. Define $\mathcal{I}^+$ by integrating $\sum_\iota \max(\mathcal{I}_\iota,0)+\max(J,0)$ over $(0,1)\times\mathbb{R}^3$, and define $\mathcal{I}^-$ using $\max(-\mathcal{I}_\iota,0)$ and $\max(-J,0)$. The positive or negative part is taken after forming each complete $\mathcal{I}_\iota$ and the complete J. These integrals lie in $[0,\infty]$. When $\mathcal{I}^-<\infty$, the notation $\int_0^1\int_{\mathbb{R}^3}(\sum_\iota \mathcal{I}_\iota + J)\,dy\,d\theta$ means $\mathcal{I}^+-\mathcal{I}^-$. Proposition B.1 shows that $\mathcal{I}^-<\infty$ implies $\mathcal{I}^+<\infty$ under its admissibility hypotheses. Proposition B.8 supplies the required bound on $\mathcal{I}^-$ from the cell, tail and exterior estimates.

For $\alpha\in\mathcal{K}$, the weights are *admissible* at α if, besides the regularity and decay ensured by their form, the transport and reference bounds hold at every time and point, and each Ω_p has a positive lower bound on $\mathcal{N}_p$. One also requires $\mathsf{I}'_{2,\iota}\ge 0$ for every ι and the following bounds with given numbers $\mathsf{n}_\lambda, \mathsf{n}_{\nabla\lambda}$ and $\mathsf{n}_\mu\ge 0$:

$$\begin{gathered}\sum_{p\in\mathcal{P}_\lambda}\lambda_p\le \mathsf{n}_\lambda,\qquad \sum_{p\in\mathcal{P}_\lambda}|\nabla\lambda_p|\le \mathsf{n}_{\nabla\lambda},\\ \Big(\sum_{p\in\mathcal{P}_\mu}|\operatorname{sgn}_p(\iota)|\,\mu_p\Big)^2\le \mathsf{n}_\mu\sum_{p\in\mathcal{P}_\mu}|\operatorname{sgn}_p(\iota)|\,|\Omega_p(\iota)|\,\mu_p\end{gathered}\tag{B.19}$$

These inequalities hold at every point and, for the last one, every ι. Admissibility is required for every $\alpha\in\mathcal{K}$, with one common triple of bounds.

Proposition B.1. *For data of the form of Subsection B.1 and weights of the form above, let $\alpha\in\mathcal{K}$, and let the weights be admissible at α. Let $v\in\mathcal{D}^\Gamma_{S(\alpha)}$, $\beta\in\mathbb{R}^n$ and $F\in\mathcal{H}$ be such that $F\simeq \mathcal{T}_{S(\alpha)}v+\mathcal{C}^{\mathrm{r}}_\alpha v+\sum_i\beta_i b_i$. Then $\|v\|_\mu<\infty$, the field $\mathsf{M}v\in\mathcal{H}_{\mathrm{div}}$ of (B.14) exists, and*

$$\|\mathsf{M}v\|_{\mathcal{H}}\le c_{\mathsf{M}}\Big(\tfrac{3}{50}\,\|v\|_w^2+c_\mu\,\|v\|_\mu^2\Big)^{1/2},\qquad c_{\mathsf{M}}^2=\tfrac{50}{3}\,\big(1+\mathsf{n}_\lambda+\mathsf{n}_{\nabla\lambda}/\sqrt{w_*}\big)^2+\mathsf{n}_\mu/c_\mu.$$

If, moreover, $\mathcal{I}^-<\infty$, then $\mathcal{I}^+<\infty$ and

$$\langle F,\mathsf{M}v\rangle_{\mathcal{H}}-\tfrac{3}{50}\,\|v\|_w^2-c_\mu\,\|v\|_\mu^2-\sum_i\beta_i\,\langle b_i,\mathsf{M}v\rangle_{\mathcal{H}}\ge\int_0^1\int_{\mathbb{R}^3}\Big(\sum_\iota\mathcal{I}_\iota+J\Big)\,dy\,d\theta.\tag{B.20}$$

Proof sketch. Write $S=S(\alpha)$ and $h(y)=(1+|y|)^{-1}$. Hardy's inequality gives $hv\in\mathcal{H}$. Multiplying $S^{-1}\partial_\theta v=\mathcal{T}_S v+\Delta v-\frac12 v-\frac12 y\cdot\nabla v$ by h puts every term on the right in $\mathcal{H}$. The angular derivative in the rotating vector frame has size at most $|y|\,|\nabla v|+|v|$, so it too lies in $\mathcal{H}$ after multiplication by h. Parseval in time and angle now bounds the sum of the squared time and angular frequencies times $\|h\hat v_\iota\|^2_{\mathcal{H}}$. The weights satisfy $\mu_p\le Ch^2$, and (B.15) is linear in the two integer frequencies. Since $|j|\le j^2$ for integers j, this gives $\|v\|_\mu<\infty$.

Integration by parts in the curl term of (B.14), (B.19), and Parseval's identity bound its functional of v' by $((1+\mathsf{n}_\lambda)\|\Delta v\|_{\mathcal{H}}+\mathsf{n}_{\nabla\lambda}\|\nabla v\|_{\mathcal{H}}+\sqrt{\mathsf{n}_\mu}\|v\|_\mu)\|v'\|_{\mathcal{H}}$. Hilbert-space representation thus gives $\mathsf{M}v$. By (7.14), $\|\Delta v\|_{\mathcal{H}}\le\|v\|_w$ and $\|\nabla v\|_{\mathcal{H}}\le\|v\|_w/\sqrt{w_*}$; Cauchy–Schwarz in these two resulting terms gives the stated c_{M}. The relation $\simeq$ in (7.2) permits replacing $F-\sum_i\beta_i b_i$ by $\mathcal{T}_S v+\mathcal{C}^{\mathrm{r}}_\alpha v$ when paired with the solenoidal $\mathsf{M}v$. For smooth perturbations, (7.43) and an integration by parts in the terms with W^{r} give

$$\langle\mathcal{T}_S v+\mathcal{C}^{\mathrm{r}}_\alpha v,-\Delta v\rangle_{\mathcal{H}}=\int_0^1\int_{\mathbb{R}^3}\big(|\nabla\operatorname{curl}v|^2+\tfrac14\,|\operatorname{curl}v|^2+J\big)\,dy\,d\theta.$$

For solenoidal C^2 fields u with $\nabla u, u/(1+|y|)\in L^2$, integration by parts twice gives $\int \mathsf{Z}_\iota(|\nabla u|^2-|\operatorname{curl}u|^2)=\int\nabla^2\mathsf{Z}_\iota[u,u]$. The global C^2 weight has bounded $(1+|y|)^j|\nabla^j\mathsf{Z}_\iota|$, $j\le 2$, allowing removal of spatial cutoffs. Apply this to each mode's real and imaginary parts, and with D_ι to their derivatives. Operator-norm bounds on the weight Hessians give the baseline coefficients of (B.16). Pair the skew terms with the curl of the equation for a λ

index and with its Leray projection for a μ index. In the rotating frame, the time derivative supplies $i\Omega_p(\iota)$. The sign on $\mathcal{N}_p$ makes the contribution positive; the μ terms give $\|v\|_\mu^2$, retaining the fraction c_μ on the left of (B.20). The remaining products come from transport by $W^{\mathrm{r}} + \frac{1}{2}y - \nu_p \mathbf{e}_z \times y$, coupling by reference harmonics, rotation, and derivatives of the weights in the Laplacian terms.

Approximation in the domain of $\mathcal{T}_S$. Extend v periodically and set $v_\delta = \eta_\delta * v$ for a smooth probability kernel supported in $|\tau| + |z| \le \delta$, radial in z. Convolution preserves period and divergence; changing variables $z \mapsto Rz$ in (7.5) preserves Γ-invariance. Translation continuity gives $\nabla v_\delta \to \nabla v$ and $\nabla^2 v_\delta \to \nabla^2 v$ in $\mathcal{H}$, since Plancherel and (7.4) give $\nabla^2 v \in \mathcal{H}$. The Hardy inequality used in Lemma B.11 then gives $(v_\delta - v)/(1+|y|) \to 0$ in $\mathcal{H}$.

The only commutator in $\mathcal{T}_S v_\delta - \eta_\delta * (\mathcal{T}_S v)$ is $\frac{1}{2}\int \eta_\delta(\tau, z)\, z \cdot \nabla v(\theta - \tau, y - z)\, d\tau\, dz$. Its $\mathcal{H}$ norm is at most $\delta \|\nabla v\|_{\mathcal{H}}/2$, so $\mathcal{T}_S v_\delta \to \mathcal{T}_S v$. This argument never takes a separate global norm of $\partial_\theta v$ or $y \cdot \nabla v$. The formal approximation lemma records the higher integrability of v_δ used in the smooth identities.

The C^2 joins of $\mathsf{D}_\iota, \mathsf{Z}_\iota$ supply matching traces for two integrations by parts. The globally Lipschitz λ_p, μ_p have the first weak derivatives required in the skew terms. Their first distributional derivatives have no surface-measure contribution at $y_3 = 0$ because the weights are continuous there. Their derivative bounds and the weighted L^2 estimates permit removal of spatial cutoffs. Negative error terms of the intermediate decomposition converge; Fatou's lemma applies to its nonnegative terms along a subsequence. Young's inequality $ab \le (\varepsilon a^2 + \varepsilon^{-1} b^2)/2$, with $\varepsilon > 0$ fixed by the auxiliary data, distributes the products among the coefficients of (B.16). If $\mathcal{I}^- < \infty$, the resulting bound gives $\mathcal{I}^+ < \infty$ and (B.20).

The forcing harmonics avoid $\pm\mathcal{N}_p$, so their skew pairings in (B.14) vanish; only the pairing with $-\Delta v$ remains. For every v, β, F as in Proposition B.1, it remains to prove $\mathcal{I}^- < \infty$ and

$$\int_0^1 \int_{\mathbb{R}^3} \Big(\sum_\iota \mathcal{I}_\iota + J \Big)\, dy\, d\theta \ge \tfrac{3}{50}\, \omega^2 |\beta|^2 - \sum_i \beta_i \, \langle b_i, \mathsf{M} v \rangle_{\mathcal{H}} - \tfrac{3}{50}\, [F]^2 - 1000\, |\ell(v)|^2. \tag{B.21}$$

Subsection B.3 derives (B.21) from the cell and assembly conditions. Choose c_1, c_2, c_0 with

$$\Big(\tfrac{50}{3}\Big)^{1/2} c_{\mathsf{M}} \le c_1, \qquad \Big(1000 \cdot \tfrac{50}{3}\Big)^{1/2} \le c_2, \qquad \tfrac{50}{3} \cdot 400\,(1 + 2^{-18}) \le c_0^2. \tag{B.22}$$

As $c_0 \ge 0$ in (7.17), the bound $\|e_k\|_{\mathcal{H}}^2 = \tau_k \le \frac{50}{3} 400(1 + 2^{-18}) \le c_0^2$ from (B.11) and (B.22) gives $\|e_k\|_{\mathcal{H}} \le c_0$, as required.

Lemma B.2. *Let α, v, β and F be as in Proposition B.1, and suppose that $\mathcal{I}^- < \infty$ and that (B.21) holds. Then (7.18) holds for v, β and F, with these numbers c_1 and c_2.*

Proof. By Proposition B.1 and the assumed $\mathcal{I}^- < \infty$, both $\mathcal{I}^\pm$ and $\|v\|_\mu$ are finite. Adding (B.20) and (B.21) cancels the real density integral and the forcing pairings. With (7.14),

$$P^2 \le \langle F, \mathsf{M} v \rangle_{\mathcal{H}} + Q^2, \qquad P^2 = \tfrac{3}{50}\, \|(v, \beta)\|_{w,\omega}^2 + c_\mu \, \|v\|_\mu^2, \qquad Q^2 = \tfrac{3}{50}\, [F]^2 + 1000\, |\ell(v)|^2,$$

with $P, Q \ge 0$. Cauchy–Schwarz and the multiplier bound give $\langle F, \mathsf{M} v \rangle_{\mathcal{H}} \le aP$, where $a = c_{\mathsf{M}} \|F\|_{\mathcal{H}} \ge 0$, because $\frac{3}{50}\|v\|_w^2 + c_\mu \|v\|_\mu^2 \le P^2$. Completing the square in $P^2 \le aP + Q^2$ gives $(P - \frac{1}{2}a)^2 \le Q^2 + \frac{1}{4}a^2 \le (Q + \frac{1}{2}a)^2$. Hence $P \le a + Q$. Using $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$ for $x, y \ge 0$ in the definition of Q, we obtain

$$P \le a + Q \le c_{\mathsf{M}} \|F\|_{\mathcal{H}} + \big(\tfrac{3}{50}\big)^{1/2} [F] + 1000^{1/2} |\ell(v)|.$$

Since $\sqrt{3/50}\, \|(v, \beta)\|_{w,\omega} \le P$, division by $\sqrt{3/50} > 0$ and the first two bounds of (B.22) give (7.18). The coefficient of $[F]$ is one. □

The common $c_\mu, w_*, \mathsf{n}_\lambda, \mathsf{n}_{\nabla\lambda}, \mathsf{n}_\mu$ give a common c_{M} in Proposition B.1, hence the same c_1, c_2 for every $\alpha \in \mathcal{K}$, as a certificate requires.

B.3. The cells, their complement and the tree. Fix $\alpha \in \mathcal{K}$ and an arbitrary triple (v, β, F) as in (7.18). On each cell finitely many harmonics contribute coordinates to a quadratic form. Estimates for the other harmonics, the polynomial tails of the resolved harmonics, and the exterior of the cells account for the rest of the integral. Consider first the angular mean on one interior cell. Integrating the divergence equation says that the four outward fluxes sum to zero. Cells sharing a side use the same flux with opposite signs. The calculation after Proposition B.4 gives the corresponding identity for nonzero angular frequency, where an additional volume moment appears, and derives the pressure pairing. The left-hand side of (B.21) is first written on the quarter plane $\{r > 0, z > 0\}$, which is then divided into the cells of Subsection B.1 and their complement. By (B.13), and since the coefficients of $\mathcal{I}_\iota$ are invariant under the rotations about the y_3 axis, $\mathcal{I}_\iota$ is a function of (r, z) with $r > 0$ and $z = y_3 \in \mathbb{R}$; on the quarter plane we take it, and the modes, at $\theta = 0$ and at the point $(r, 0, z)$. Let $\bar{J}(r, z)$ be the mean of $J(\theta, R_\phi(r, 0, z))$ over $\theta \in (0, 1)$ and $\phi \in (0, 2\pi)$.

Lemma B.3. *Suppose that the symmetry (B.17) holds and that $\mathcal{I}^- < \infty$. Then*

$$\int_0^1 \int_{\mathbb{R}^3} \Big(\sum_\iota \mathcal{I}_\iota + J \Big)\, dy\, d\theta = 4\pi \int_0^\infty \int_0^\infty \Big(\sum_\iota \mathcal{I}_\iota + \bar{J} \Big)\, r\, dr\, dz, \tag{B.23}$$

and the negative parts of the terms on the right have, in sum, a finite integral. So do the positive parts, if $\mathcal{I}^+ < \infty$.

For J the sign is taken before averaging, giving $\overline{J^+}$ and $\overline{J^-}$. Subtract the two integral identities when $\mathcal{I}^- < \infty$.

Proof. In cylindrical coordinates the integral of $\mathcal{I}_\iota$ over $(0, 1) \times \mathbb{R}^3$ is 2π times its integral over $\{r > 0, z \in \mathbb{R}\}$ with the weight r, and that of J is 2π times that of $\bar{J}$. The pair $(R_*, 0)$ belongs to Γ, and $R_\phi R_* = R_* R_{-\phi}$. With the substitution $\phi' \mapsto -\phi'$ in (B.12) and the relation $E_\iota(\theta, -\phi) = E_{\iota^\flat}(\theta, \phi)$ this gives

$$\hat{v}_\iota(\theta, R_* y) = R_*\, \hat{v}_{\iota^\flat}(\theta, y). \tag{B.24}$$

Combining (B.24) with the coefficient symmetry (B.17) gives $\mathcal{I}_\iota(r, -z) = \mathcal{I}_{\iota^\flat}(r, z)$. Reindexing by the involution $\iota \mapsto \iota^\flat$ therefore identifies the summed integrals on $\{z < 0\}$ and $\{z > 0\}$. The term J is formed from v and W^{r}, which are invariant under $(R_*, 0)$, by the gradient, the curl, and the scalar and the vector product, which commute with the rotation R_*. Hence $J(\theta, R_* y) = J(\theta, y)$ and $\bar{J}(r, -z) = \bar{J}(r, z)$. All this holds for the positive and for the negative parts of the terms, with the means of the positive and of the negative part of J. □

The stretching term is bounded from below mode by mode. Let $\mathbb{V} = \nabla \operatorname{curl} W^{\mathrm{r}}$, and for $(\mathbf{G}, \mathbf{v}) \in \mathbb{C}^{3\times 3} \times \mathbb{C}^3$, which stands for the gradient and the value of a mode at a point, let

$$\mathcal{L}(\mathbf{G}, \mathbf{v}) = \mathsf{s}_1\, |\operatorname{curl} \mathbf{G}|^2 + \sqrt{\tfrac{7}{2}}\, \mathsf{s}_2\, |\mathbf{G}|\, |\mathbf{v}| + \tfrac{1}{2}\sqrt{\tfrac{7}{2}}\, \mathsf{s}_3\, (\mathsf{h}\, |\mathbf{G}|^2 + |\mathbf{v}|^2/\mathsf{h}). \tag{B.25}$$

Here $\operatorname{curl} \mathbf{G}$ is the vector $(\mathbf{G}_{23} - \mathbf{G}_{32}, \mathbf{G}_{31} - \mathbf{G}_{13}, \mathbf{G}_{12} - \mathbf{G}_{21})$, which is the curl of the mode when $\mathbf{G}$ is its gradient, and s_1, s_2, s_3 and $\mathsf{h} > 0$ are functions of (r, z), given on each cell and on the complement of the cells: at every point, for all times and angles, s_1 is an upper bound for the quadratic form of $\mathbb{S}$, s_2 for the operator norm of the mean $\bar{\mathbb{V}}$ of $\mathbb{V}$ over time and angle, taken in the frame $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$, and s_3 for the operator norm of $\mathbb{V} - \bar{\mathbb{V}}$ in this frame. Thus s_2 and s_3 are nonnegative, and so is s_1, because the trace of $\mathbb{S}$ vanishes.

In each cell Q there are three finite sets of harmonics, all closed under $\iota \mapsto -\iota$. The first consists of the *resolved* harmonics. The second is the subset represented by triples φ on Q in Subsection B.1. The third contains the *eliminated* harmonics. They have $\iota_1 \neq 0$, are disjoint from the resolved set, and enter the estimate through bounds without adding coordinates. All remaining harmonics, including infinitely many outside these finite sets, satisfy the inequalities after (B.28). On a cell that meets the axis only harmonics with $\iota_1 = 0$ are resolved. Each resolved harmonic of a cell has two degrees $\mathsf{d}_0 \geq 1$ and $\mathsf{d} \geq \mathsf{d}_0 + 2$, with $\mathsf{d}_0 \geq 2$ if it is the harmonic of a triple φ on Q, and the polynomials φ_r, φ_ϕ, φ_z of such a triple have degrees at most $(\mathsf{d}_0, \mathsf{d}_0 - 1)$, $(\mathsf{d}_0, \mathsf{d}_0)$ and $(\mathsf{d}_0 - 1, \mathsf{d}_0)$ in (r, z).

Each cell carries polynomial models of W^{r}, $\operatorname{curl} W^{\mathrm{r}}$, $\mathbb{S}$ and $\mathbb{V}$, represented as Ξ_i is in (B.10), together with bounds for their errors. It also carries an approximation to $1/S(\alpha)$ whose error bound holds throughout $\mathcal{K}$.

For each resolved harmonic the cell has finitely many unknowns, linear functionals of one mode of a perturbation and of its pressure in the cell: the coefficients of its gradient and of its components in products of orthonormal polynomials of degree at most d and $\mathsf{d} - 1$ in each variable, moments of its traces and of the pressure on the sides, and values at corners. At an off-axis corner the velocity coordinates are the three components of a resolved mode. There is also a stream-function coordinate for the angular mean. The formal definitions recover the velocity trace from cell integrals of the mode and its first and mixed derivatives, so changing an almost-everywhere representative at the corner has no effect. Pressure enters through side moments. To these are added $\beta \in \mathbb{R}^n$, a vector $\zeta \in \mathbb{R}^n$ and finitely many further unknowns. A trace of the perturbation or its associated pressure can occur in two neighbouring cells, so their local unknowns cannot be chosen independently. We use one real vector $x_{\mathrm{g}} = (x_{\mathrm{p}}, x_{\mathrm{f}})$. The perturbation and the given coefficients β fix x_{p}. The free coordinates x_{f} include ζ and will be chosen in the proof of Proposition B.8. The matrix X_Q reads off the unknowns for cell Q from this one vector. The matrices X_Q are formed by relations that hold for every perturbation, such as the continuity of a trace across a side, the symmetry at the plane $z = 0$, or the divergence equation on a side. The vectors of unknowns of a cell are complex, and $\langle \mathsf{A}x, x\rangle$ denotes the real part of the form of a matrix A on a vector x.

A cell has a matrix A_Q on its unknowns and, for each resolved harmonic, two numbers $\underline{\mathsf{I}}_2 > 0$ and $\underline{\mathsf{I}}'_2 \geq 0$ and a bounded measurable field $A^{\mathsf{Q},\iota}(r, z)$ of Hermitian 12×12 matrices, which act on the pairs $(\mathbf{G}, \mathbf{v})$. On the cell, $\mathsf{I}_{2,\iota} \geq \underline{\mathsf{I}}_2$ and $\mathsf{I}'_{2,\iota} \geq \underline{\mathsf{I}}'_2$. The form of $A^{\mathsf{Q},\iota}(r, z)$ on a pair $(\mathbf{G}, \mathbf{v})$ is at most the sum of the other three terms of $\mathcal{I}_\iota$ minus a form with constant coefficients, which bounds the errors of the polynomial models; and, up to forms whose coefficients are constants or constants divided by r, it is at least $\mathcal{L}(\mathbf{G}, \mathbf{v})$. For every $x = \mathsf{X}_\mathsf{Q} x_{\mathrm{g}}$, the cell matrix A_Q must satisfy the following comparison. Its forms use the cell blocks and the assembly and pressure conditions of the full specification.

$$\langle (\mathsf{A}_\mathsf{Q} - \mathsf{E}_\mathsf{Q})\, x, x\rangle \leq \langle (\mathsf{P}_\mathsf{Q} + \mathsf{B}_\mathsf{Q} - \mathsf{T}_\mathsf{Q})\, x, x\rangle + \mathsf{b}_\mathsf{Q}(x) + \mathsf{e}_\mathsf{Q}(x). \tag{B.26}$$

In (B.26), E_Q uses $\underline{\mathsf{I}}_2$, $\underline{\mathsf{I}}'_2$ and the cell-polynomial integrals of $A^{\mathsf{Q},\iota}$. The models of $\mathbb{S}$ and $\mathbb{V}$ bound P_Q above by the resolved contribution of J less the terms assigned to unresolved harmonics. The resolved-gradient pairings with harmonics of ∇b_i and Ξ_i, weighted by β_i and $2\zeta_i$ respectively, bound B_Q above. The form T_Q uses the highest-degree gradient coefficients. The eliminated harmonics, which have no unknowns, give b_Q. Pairing the right-hand side with the fields of the triples φ gives e_Q. The component of a resolved mode that is orthogonal to the polynomials of degree at most d is controlled by $\underline{\mathsf{I}}_2$: the terms that it has to absorb are at

most

$$\underline{\mathsf{l}}_2\Big[r_1 \min(\Lambda^r_{\mathsf{d}+1}, \Lambda^z_{\mathsf{d}+1}) + \frac{(|k_{\mathrm{rot}}\,\iota_1| - 2)^2_+}{r_2}\Big], \tag{B.27}$$

with 1 in place of the factor r_1 if $r_1 = 0$. Here $\Lambda^r_j = j(j+1)\,(2/(r_2 - r_1))^2$, or $j(j+2)\,(2/r_2)^2$ on a cell that meets the axis, and $\Lambda^z_j = j(j+1)\,(2/(z_2 - z_1))^2$ are the eigenvalues of the Legendre and of the Jacobi $(0,1)$ operators on the intervals of the cell [77, Tables 18.3.1 and 18.8.1]. In the radial interval touching the axis, $x = 2r/r_2 - 1$ makes $r\,dr$ a constant multiple of $(1+x)\,dx$, the Jacobi weight with parameters $(0,1)$. The eigenvalues are therefore $j(j+2)$ before rescaling; the Legendre values are $j(j+1)$. The factor from an interval of length L is $(2/L)^2$ [77, Table 18.3.1, Jacobi and Legendre rows; Table 18.8.1, rows 1 and 7]. For each harmonic that it does not resolve, a cell has a constant $\mathsf{c}^* \geq \sqrt{7/2}$ and three bounded measurable functions a'_1, a_1 and a_0 of (r,z); the complement of the cells has one such constant and three such functions for all harmonics, and there the three functions are nonnegative. In a cell that does not resolve ι, and on the complement of the cells, we let

$$\mathcal{I}^\circ_\iota = \mathcal{I}_\iota - \mathcal{L}^\circ(\nabla\hat{v}_\iota, \hat{v}_\iota) - \mathsf{a}'_1\,|\operatorname{curl}\hat{v}_\iota|^2 - \mathsf{a}_1\,|\nabla\hat{v}_\iota|^2 - \mathsf{a}_0\,|\hat{v}_\iota|^2, \tag{B.28}$$

where $\mathcal{L}^\circ$ is (B.25) with c^* in place of $\sqrt{7/2}$.

Near the plane $y_3 = 0$ there may be finitely many rectangles $\Pi = [r_1, r_2] \times [0, z_2]$ with $r_1 > 0$ in the union of the cells. Each rectangle has a harmonic ι resolved in the cells along a segment of $z = 0$; neither $\iota^\flat$ nor $-\iota^\flat$ is resolved or eliminated in any cell. Its matrix C_Π acts on moments, along these plane segments, of the $\mathbf{e}_z$ component of $\hat{v}_\iota$. The vector $\mathsf{X}_\Pi x_{\mathrm{g}}$ selects these moments from the cell vectors $\mathsf{X}_\mathsf{Q} x_{\mathrm{g}}$. The conditions on a rectangle are a bound, by a positive definite matrix, for a weighted Dirichlet integral over Π of the combinations of finitely many given functions, which vanish on the three other sides of Π and whose traces on the plane, multiplied by r, are combinations of the polynomials of the moments, and a comparison of C_Π with the inverse of this matrix.

A harmonic that is not resolved in a cell, and every harmonic on the complement of the cells, is subject to a condition on scalars at every point, which bears on the form (B.28). For each cell and each harmonic that is not resolved in it, and for each harmonic on the complement of the cells, we are given finitely many regions of the quarter plane, each with a set of harmonics, possibly infinite, which contains the harmonic for which the region is given, with a number k_0 such that $|k_{\mathrm{rot}}\iota_1| \geq k_0$ on this set, with lower bounds $\underline{\mathsf{l}}_2$, $\underline{\mathsf{l}}'_2$, $\underline{\mathsf{l}}'_1$, $\underline{\mathsf{l}}_1$ and $\underline{\mathsf{l}}_0$ for the five coefficients of $\mathcal{I}_\iota$ which hold on the region for all the harmonics of the set, and with a function $\mathsf{t} \geq 0$ or, if the region reaches infinity, with the four numbers of (B.30). The regions of a cell and a harmonic cover the cell; those of a harmonic on the complement lie in the complement, cover it and overlap only in sets of measure zero. On a region, with the constant c^* and the three functions of the cell and the harmonic for which it is given, or of the complement, let

$$\mathsf{f}_2 = \underline{\mathsf{l}}_2, \quad \mathsf{f}'_2 = \underline{\mathsf{l}}'_2, \quad \mathsf{f}'_1 = \underline{\mathsf{l}}'_1 - \mathsf{s}_1 - \mathsf{a}'_1, \quad \mathsf{f}_1 = \underline{\mathsf{l}}_1 - \tfrac{1}{2}\mathsf{c}^*\mathsf{s}_3\,\mathsf{h} - \mathsf{a}_1, \quad \mathsf{f}_0 = \underline{\mathsf{l}}_0 - \frac{\mathsf{c}^*\mathsf{s}_3}{2\,\mathsf{h}} - \mathsf{a}_0, \quad \mathsf{f}_{01} = \mathsf{c}^*\mathsf{s}_2,$$

and

$$\mathsf{f}^*_1 = \mathsf{f}_1 + 2\min\big(\mathsf{f}'_1 + \mathsf{f}'_2\,(k_0 - 1)^2_+/r^2,\, 0\big) + \mathsf{f}_2\,(k_0 - 2)^2_+/r^2, \qquad \mathsf{f}^*_0 = \mathsf{f}_0 + \mathsf{t}\,(k_0 - 1)^2_+/r^2.$$

The condition is that, at every point of the region,

$$\mathsf{f}_2 \geq 0, \qquad 0 \leq \mathsf{t} \leq \mathsf{f}^*_1, \qquad \mathsf{f}^*_0 \geq 0, \qquad \mathsf{f}^2_{01} \leq 4\,(\mathsf{f}^*_1 - \mathsf{t})\,\mathsf{f}^*_0. \tag{B.29}$$

Some of the regions on the complement of the cells reach infinity: they are sets on which $|(r,z)|$ exceeds a radius, which may depend on the direction, or, for harmonics with $\iota_1 = 0$, sets $\{r > a \text{ or } z > a\}$ with a given number $a > 0$. On these the condition is not pointwise.

We require $\mathsf{f}_2 \geq 0$, $\mathsf{f}_1^* \geq \mathsf{f}_1^\infty$, $|(r,z)|^2 \max(-\mathsf{f}_0, 0) \leq \mathsf{f}_0^\infty$ and $|(r,z)|\,\mathsf{f}_{01} \leq \mathsf{f}_{01}^\infty$, with numbers such that

$$\frac{\mathsf{f}_0^\infty}{\Lambda_0} + \frac{\mathsf{f}_{01}^\infty}{\sqrt{\Lambda_0}} \leq \mathsf{f}_1^\infty, \qquad 0 < \Lambda_0 \leq \tfrac{1}{4} + (k_0 - 1)_+^2 \quad \text{or} \quad 0 < \Lambda_0 \leq 2 \quad \text{respectively,} \tag{B.30}$$

and the form is integrated over the region Z, with the inequality

$$\Lambda_0 \int_Z \frac{|\hat{v}_\iota|^2}{|(r,z)|^2}\, r\, dr\, dz \leq \int_Z |\nabla \hat{v}_\iota|^2\, r\, dr\, dz. \tag{B.31}$$

Proposition B.4. *Suppose that the data satisfy the conditions of this appendix. Then $\mathcal{I}^- < \infty$, and at almost every point of the complement of the cells in the quarter plane*

$$\bar{J} \geq -\sum_\iota \mathcal{L}(\nabla \hat{v}_\iota, \hat{v}_\iota), \tag{B.32}$$

where the sum is over the harmonics (B.2). If, moreover, $\mathcal{I}^+ < \infty$, then there is a vector x_{p} of unknowns of the first set, which contains β and is zero if v, β and F are zero, with the following properties. For every x_{f}, with $x_{\mathrm{g}} = (x_{\mathrm{p}}, x_{\mathrm{f}})$ and with the entries ζ_i of x_{f}, there are real numbers $\mathsf{r}_{\mathsf{Q},0}, \ldots, \mathsf{r}_{\mathsf{Q},3}$ such that, for every cell Q,

$$\langle \mathsf{A}_\mathsf{Q} \mathsf{X}_\mathsf{Q} x_{\mathrm{g}}, \mathsf{X}_\mathsf{Q} x_{\mathrm{g}} \rangle + \mathsf{r}_{\mathsf{Q},0} + \sum_\iota \int_\mathsf{Q} \mathcal{I}_\iota^\circ\, r\, dr\, dz \leq \int_\mathsf{Q} \Big(\sum_\iota \mathcal{I}_\iota + \bar{J} \Big)\, r\, dr\, dz + \mathsf{r}_{\mathsf{Q},1} + \mathsf{r}_{\mathsf{Q},2} + \mathsf{r}_{\mathsf{Q},3} |\beta|^2, \tag{B.33}$$

where the sum on the left is over the harmonics that are not resolved in Q and converges absolutely, and

$$\begin{aligned} &\sum_\mathsf{Q} \big(\mathsf{r}_{\mathsf{Q},1} + \mathsf{r}_{\mathsf{Q},2} + \mathsf{r}_{\mathsf{Q},3}\, |\beta|^2 - \mathsf{r}_{\mathsf{Q},0} \big) + 2 \sum_\Pi \langle \mathsf{C}_\Pi \mathsf{X}_\Pi x_{\mathrm{g}}, \mathsf{X}_\Pi x_{\mathrm{g}} \rangle \\ &\qquad \leq \frac{1}{4\pi} \Big(\tfrac{3}{50}\, [F]^2 + \sum_i \beta_i \langle b_i, \mathsf{M} v \rangle_{\mathcal{H}} + 2\, \zeta \cdot \ell(v) \Big) + \mathsf{n}_b\, |\beta|^2, \end{aligned} \tag{B.34}$$

where the second sum is over the rectangles and n_b is a given number. Finally, for every region $Z = \{r > a \text{ or } z > a\}$ and every harmonic ι of its set, the inequality (B.31) holds with $\Lambda_0 = 2$.

Proof sketch. The finiteness of $\mathcal{I}^-$ follows from lower bounds for the coefficients of $\mathcal{I}_\iota$, which are among the conditions, and, for J, from the bounds for the derivatives of W^{r}, with Hardy's inequality. To obtain (B.32), expand v and ∇v in time and angle at each (r,z). Parseval's identity expresses the means of their products through the modes. The first term in (B.18) is bounded below by $-\mathsf{s}_1\, |\operatorname{curl} v|^2$. The other two terms form a bilinear expression in v and ∇v.

For a trace-zero matrix in place of $\mathbb{V}$, this bilinear expression has absolute value at most $\sqrt{7/2}$ times the matrix's operator norm times $|v|\,|\nabla v|$. Apply the bound to $\bar{\mathbb{V}}$ mode by mode, and to $\mathbb{V} - \bar{\mathbb{V}}$ at each time and angle. Young's inequality then gives the last term of (B.25).

In each cell, $\mathsf{X}_\mathsf{Q} x_{\mathrm{g}}$ records the mode and pressure unknowns determined by the equation, together with x_{f}, up to a vector that leaves the quadratic form of A_Q unchanged. Decompose each resolved mode into its polynomial part, whose coefficients are unknowns, and its orthogonal remainder. By the first bound for $A^{\mathsf{Q},\iota}$, the integral of the term $\mathcal{I}_\iota$ of a resolved harmonic is at least that of the form of $A^{\mathsf{Q},\iota}$ on the mode plus $\underline{\mathsf{l}}_2$ and $\underline{\mathsf{l}}_2'$ times the parts of the integrals of $|\nabla^2 \hat{v}_\iota|^2$ and $|\nabla \operatorname{curl} \hat{v}_\iota|^2$ that the unknowns do not determine. In the expansion of $\bar{J}$ a harmonic of $\mathbb{S}$ or of $\mathbb{V}$ couples two modes whose harmonics differ by it. The couplings between the polynomials of resolved modes are computed with the polynomial models. An eliminated mode is removed pointwise, through its angular derivative, which requires $\iota_1 \neq 0$. Here $k_{\mathrm{rot}} \geq 3$ by Subsection B.1, so $k = k_{\mathrm{rot}} \iota_1$ has $|k| \geq 3$. In particular $k \neq 0$ and $k^2 \neq 1$, as required for elimination. All other couplings are bounded, as in (B.32), by the expressions

(B.25) for the remainders and for the modes that are not resolved, which the second bound for $A^{\mathsf{Q},\iota}$ and the differences $\mathcal{I}_\iota - \mathcal{I}_\iota^\circ$ absorb. The pairings of F with the fields of the triples φ are computed from the equation by an integration by parts over the cell; $\mathsf{r}_{\mathsf{Q},1}$ is a form of these pairings, whose sum over the cells is $\frac{3}{50}[F]^2/4\pi$, and $\mathsf{r}_{\mathsf{Q},2}$ is the pairing over Q of the gradients of the modes with the harmonics of ∇b_i and of Ξ_i, up to errors bounded by $\mathsf{r}_{\mathsf{Q},3}|\beta|^2$, by $\mathsf{n}_b|\beta|^2$ and by a part of $\mathsf{r}_{\mathsf{Q},0}$. The parts of the Hessian and gradient-of-curl integrals that the unknowns do not determine, with coefficients $\underline{\mathsf{l}}_2$ and $\underline{\mathsf{l}}'_2$, absorb the remainders through (B.27), together with T_Q. Applying the matrix comparison (B.26) now gives (B.33). The other part of $\mathsf{r}_{\mathsf{Q},0}$ belongs to the rectangles: the mode $\hat{v}_{\iota^\flat}$ is divergence-free, and its component along $\mathbf{e}_z$ on the plane $z = 0$ is minus that of $\hat{v}_\iota$, by (B.24), so that an integration by parts over Π and the conditions on the rectangle bound the form of $2\,\mathsf{C}_\Pi$ on the moments of this trace by the integral over Π of $|\hat{v}_{\iota^\flat}|^2 + |\hat{v}_{-\iota^\flat}|^2$ times a function which is part of the function a_0 of these harmonics. The inequality (B.34) follows by adding over the cells.

One interior cell. On an interior cell $\mathsf{Q} = [r_1, r_2] \times [z_1, z_2]$ with $r_1, z_1 > 0$, write a resolved mode as $E_\iota(u_r, u_\phi, u_z)$ in the cylindrical frame. Its divergence equation is $\partial_r(ru_r) + ik_{\mathrm{rot}}\iota_1 u_\phi + r\partial_z u_z = 0$. For the constant test polynomial, put $k = k_{\mathrm{rot}}\iota_1$ and integrate over Q. The resulting moment relation is

$$\int_{z_1}^{z_2} [ru_r]_{r_1}^{r_2}\, dz + \int_{r_1}^{r_2} r[u_z]_{z_1}^{z_2}\, dr + ik \int_{\mathsf{Q}} u_\phi\, dr\, dz = 0.$$

For $k = 0$ the four outward side fluxes sum to zero. For $k \neq 0$ they also determine a volume moment of u_ϕ. On a common radial side the two outward fluxes have opposite signs. The map $\mathsf{X}_\mathsf{Q} x_{\mathrm{g}}$ reads a shared trace from the same coordinates on either side. For the axial component of a momentum test, let p_ι denote the pressure coefficient and $\psi(r, z)$ the axial test polynomial. Integration by parts gives

$$\int_{\mathsf{Q}} r\, \partial_z p_\iota\, \overline{\psi} = \int_{r_1}^{r_2} r[p_\iota \overline{\psi}]_{z_1}^{z_2}\, dr - \int_{\mathsf{Q}} r\, p_\iota\, \partial_z \overline{\psi},$$

where the cell integrals use $dr\, dz$. This exhibits the pressure trace in a momentum row. For a spatially constant pressure shift the two terms it adds on the right cancel by the fundamental theorem of calculus. The assembled form must respect the pressure freedoms specified in (B.26).

For example, if a scalar entry a of a coefficient is represented by a_{mod} with $|a - a_{\mathrm{mod}}| \leq \epsilon$ on Q, the omitted cross term on two polynomial components P_1, P_2 has magnitude at most $2\epsilon \int_{\mathsf{Q}} |P_1 P_2|\, r\, dr\, dz \leq \epsilon \int_{\mathsf{Q}} (|P_1|^2 + |P_2|^2)\, r\, dr\, dz$. The constant-coefficient error form preceding (B.26) reserves this amount from the lower bound. The modeled contribution is a matrix entry obtained by integrating the two basis polynomials against $a_{\mathrm{mod}}\, r\, dr\, dz$. The first omitted degree in (B.27) contributes the smaller of the two rescaled Legendre eigenvalues times $r_1 \underline{\mathsf{l}}_2$. Angular differentiation bounds the eliminated modes; (B.28)–(B.31) cover the other unrepresented ones. Thus (B.33) bounds the actual mode, including its polynomial remainder. Proposition B.8 sums the cell inequalities using (B.34) for their shared terms and (B.32) on the exterior.

The last assertion is obtained, for each component of a mode with $\iota_1 = 0$, by completing a square with the function $r/|y|^2$, for the components along $\mathbf{e}_r$ and $\mathbf{e}_\phi$, which vanish on the axis, or $z/|y|^2$, for the component along $\mathbf{e}_z$, which vanishes on the plane $z = 0$ by (B.24), and by integrating over the region. For $a > 0$ and $g \in C_c^\infty(\mathbb{R}^2; \mathbb{R})$, put $\rho^2 = r^2 + z^2$ and integrate with measure $r\, dr\, dz$ on $Z = \{r, z > 0 : r > a \text{ or } z > a\}$. The two scalar estimates are $2\int_Z g^2/\rho^2 \leq \int_Z (g_r^2 + g_z^2)$ when $g(r, 0) = 0$, and $2\int_Z g^2/\rho^2 \leq \int_Z (g_r^2 + g_z^2 + g^2/r^2)$ when $g(0, z) = 0$. The extra term in the second estimate is the angular contribution to the vector

gradient. For modes with $\iota_1 = 0$, apply these to the real and imaginary components, then pass to the weak modes by approximation, to obtain (B.31) with $\Lambda_0 = 2$.

Lemma B.5. *Let $u = \hat{v}_\iota$ and $k = k_{\mathrm{rot}}\iota_1$. At almost every point of the quarter plane,*

$$|\operatorname{curl} u|^2 \le 2\,|\nabla u|^2, \qquad (|k|-1)_+\,|u| \le r\,|\nabla u|, \tag{B.35}$$
$$(|k|-2)_+\,|\nabla u| \le r\,|\nabla^2 u|, \quad (|k|-1)_+\,|\operatorname{curl} u| \le r\,|\nabla \operatorname{curl} u|.$$

Let the weights be admissible, and let a region be one of those given for ι, on the complement of the cells or in a cell Q that does not resolve ι. Then, at almost every point of the region, and of Q in the second case, at which $\mathsf{f}_2 \ge 0$,

$$\mathcal{I}_\iota^\circ \ge \mathsf{f}_1^*\,|\nabla u|^2 + \mathsf{f}_0\,|u|^2 - \mathsf{f}_{01}\,|\nabla u|\,|u|. \tag{B.36}$$

In particular $\mathcal{I}_\iota^\circ \ge 0$ almost everywhere on the region, and in Q in the second case, if (B.29) holds on the region.

Proof. The first inequality of (B.35) holds because the components of the curl are the differences $\partial_j u_l - \partial_l u_j$. The derivative of (B.13) with respect to ϕ_0 at $\phi_0 = 0$ is $r\,(\mathbf{e}_\phi \cdot \nabla)u = ik\,u + \mathbf{e}_z \times u$, which gives the second. In the same way the relation $\nabla u(R_\phi y) = e^{ik\phi} R_\phi\,\nabla u(y)\,R_{-\phi}$ gives that $r\,(\mathbf{e}_\phi \cdot \nabla)\nabla u - ik\,\nabla u$ is the commutator of ∇u and of the matrix of the cross product with $\mathbf{e}_z$, whose norm is at most $2\,|\nabla u|$; this gives the third inequality. The fourth is the second for $\operatorname{curl} u$, since $\operatorname{curl} u(R_\phi y) = e^{ik\phi} R_\phi \operatorname{curl} u(y)$.

Let now the region and the point be as in the statement. There $\mathcal{I}_\iota^\circ$ is formed with the constant c^* and the three functions with which the numbers f of the region are formed. The coefficients of $\mathcal{I}_\iota$ are at least the lower bounds of the region, because ι belongs to its set, and $\mathsf{l}'_{2,\iota} \ge 0$, because the weights are admissible. Since $|k| \ge k_0$, the term of $|\nabla \operatorname{curl} u|^2$ in $\mathcal{I}_\iota$ is at least $\mathsf{f}'_2\,(k_0-1)_+^2\,r^{-2}\,|\operatorname{curl} u|^2$ for either sign of f'_2. When $\mathsf{f}'_2 \ge 0$, use the last inequality of (B.35) and $\mathsf{l}'_{2,\iota} \ge \mathsf{f}'_2$. When $\mathsf{f}'_2 < 0$, the left-hand term is nonnegative by admissibility and the stated lower bound is nonpositive. Hence

$$\mathcal{I}_\iota^\circ \ge \mathsf{f}_2\,|\nabla^2 u|^2 + (\mathsf{f}'_1 + \mathsf{f}'_2\,(k_0-1)_+^2\,r^{-2})|\operatorname{curl} u|^2 + \mathsf{f}_1\,|\nabla u|^2 + \mathsf{f}_0\,|u|^2 - \mathsf{f}_{01}\,|\nabla u|\,|u|.$$

By the first inequality of (B.35) the second term on the right is at least $2\min\big(\mathsf{f}'_1 + \mathsf{f}'_2\,(k_0 - 1)_+^2\,r^{-2}, 0\big)|\nabla u|^2$, and by the third, since $\mathsf{f}_2 \ge 0$, the first is at least $\mathsf{f}_2\,(k_0-2)_+^2\,r^{-2}\,|\nabla u|^2$. This is (B.36). If (B.29) holds, then $\mathsf{t}\,|\nabla u|^2 \ge \mathsf{t}\,(k_0-1)_+^2\,r^{-2}\,|u|^2$ by the second inequality, so that $\mathcal{I}_\iota^\circ \ge (\mathsf{f}_1^* - \mathsf{t})\,|\nabla u|^2 + \mathsf{f}_0^*\,|u|^2 - \mathsf{f}_{01}\,|\nabla u|\,|u|$, a quadratic form with nonnegative coefficients of the squares and a nonpositive discriminant. □

Lemma B.6. *Let Z be a region on which $|(r,z)|$ exceeds a radius that depends on the direction, with a number Λ_0 as in (B.30), and let ι belong to its set of harmonics. Then (B.31) holds. Let now Z be a region that reaches infinity, given for ι, for which (B.31) holds. If the weights are admissible and the conditions on Z hold, then the negative part of $\mathcal{I}_\iota^\circ$ is integrable over Z with the weight r, and $\int_Z \mathcal{I}_\iota^\circ\,r\,dr\,dz \ge 0$.*

Proof. We write $u = \hat{v}_\iota$, $k = k_{\mathrm{rot}}\iota_1$ and $\rho = |(r,z)| = |y|$. The real and the imaginary part of u belong to the completion of $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ in the norm $\|\nabla\,\cdot\,\|_{L^2}$, because $v(\theta) \in X$ and (B.12) is a mean of rotated fields, and the integrals in (B.31) are those over the set of the points y with $y_3 > 0$ and $(r, y_3) \in Z$, divided by 2π. Let $U \in C_c^\infty(\mathbb{R}^3;\mathbb{C})$. On a ray from the origin, on which Z is the interval $\rho > a$,

$$\int_a^\infty |U|^2\,d\rho = -a\,|U(a)|^2 - 2\,\mathrm{Re}\int_a^\infty \rho\,\bar{U}\,\partial_\rho U\,d\rho, \quad \text{so that} \quad \int_a^\infty |U|^2\,d\rho \le 4\int_a^\infty |\partial_\rho U|^2\rho^2\,d\rho \tag{B.37}$$

by discarding the nonpositive boundary term and applying the Cauchy–Schwarz inequality. Integrate over the rays, approximate in the homogeneous H^1 norm by C_c^∞ functions, and apply Fatou's lemma to the nonnegative left-hand integral. This gives $\frac14 \int_Z |u|^2 \rho^{-2}\, r\, dr\, dz \le \int_Z |\partial_\rho u|^2\, r\, dr\, dz$. By the proof of Lemma B.5, $(|k|-1)_+\, |u| \le r\, |(\mathbf{e}_\phi \cdot \nabla) u|$, and $r \le \rho$. Since the radial direction and $\mathbf{e}_\phi$ are orthogonal, $|\partial_\rho u|^2 + |(\mathbf{e}_\phi \cdot \nabla) u|^2 \le |\nabla u|^2$. This gives (B.31) with $\frac14 + (|k|-1)_+^2 \ge \Lambda_0$ in the place of Λ_0.

For the last assertion, $\mathsf{f}_{01}^\infty \ge 0$ and $\mathsf{f}_0^\infty \ge 0$, and by (B.36), the conditions on Z and Young's inequality,

$$\mathcal{I}_\iota^\circ \ge \mathsf{f}_1^\infty\, |\nabla u|^2 - \mathsf{f}_0^\infty \frac{|u|^2}{\rho^2} - \mathsf{f}_{01}^\infty \frac{|\nabla u|\, |u|}{\rho} \ge \Big(\mathsf{f}_1^\infty - \frac{\mathsf{f}_{01}^\infty}{2\sqrt{\Lambda_0}}\Big) |\nabla u|^2 - \Big(\mathsf{f}_0^\infty + \frac{\sqrt{\Lambda_0}\, \mathsf{f}_{01}^\infty}{2}\Big) \frac{|u|^2}{\rho^2}.$$

The gradient energy is finite, and (B.31) makes the integral of $|u|^2/\rho^2$ finite. Thus the displayed lower bound is integrable and controls the negative part of $\mathcal{I}_\iota^\circ$. Using (B.31) once more, its integral is at least $(\mathsf{f}_1^\infty - \mathsf{f}_{01}^\infty/\sqrt{\Lambda_0} - \mathsf{f}_0^\infty/\Lambda_0) \int_Z |\nabla u|^2\, r\, dr\, dz$. The coefficient is nonnegative by (B.30). □

Completing the prescribed coordinates. Write the global coordinates as $E_\mathrm{p} \oplus E_\mathrm{f}$, prescribed and free. Suppose the form is nonpositive on E_f and positive on $H \setminus \{0\}$, with $\operatorname{codim} H \le \dim E_\mathrm{f}$. Since $H \cap E_\mathrm{f} = \{0\}$, projection $H \to E_\mathrm{p}$ is injective, hence onto by dimension. Each prescribed component therefore has a completion in H. Lemma B.7 takes H perpendicular to the tree columns; Proposition B.8 gets the opposite sign on E_f from the zero perturbation.

The tree combines cell bounds that may share the same unknown. When we sum the node inequalities, every nonroot node contributes its quadratic form once on either side, so these intermediate forms cancel. The root form is nonnegative. The positive pivot terms left in (B.41) give strict positivity on nonzero vectors orthogonal to the columns of the matrices Y_ξ. Besides a real matrix A_0 on the global unknowns, each node ξ carries a set $\mathcal{U}_\xi$ of pivot unknowns, positive lower bounds $\underline{\mathsf{p}}_{\xi,j}$ for $j \in \mathcal{U}_\xi$, and a matrix Y_ξ with finitely many real columns in the space of global unknowns. To the nodes are attached matrices on the global unknowns, which enter only through their forms on real vectors: the matrix A_0; the blocks into which the matrix A_Q of each cell is divided by a partition of the unknowns of the cell, carried to the global unknowns by X_Q, two blocks that are symmetric to each other being attached to the same node; twice the matrices C_Π of the rectangles, carried by X_Π; and finitely many matrices which are attached twice with opposite signs. Let N_ξ be the sum of the matrices attached to the node ξ, and let $\mathsf{N} = \sum_\xi \mathsf{N}_\xi$, so that

$$\langle \mathsf{N}\, x, x\rangle = \langle \mathsf{A}_0\, x, x\rangle + \sum_\mathsf{Q} \langle \mathsf{A}_\mathsf{Q} \mathsf{X}_\mathsf{Q} x, \mathsf{X}_\mathsf{Q} x\rangle + 2 \sum_\Pi \langle \mathsf{C}_\Pi \mathsf{X}_\Pi x, \mathsf{X}_\Pi x\rangle \tag{B.38}$$

for every real vector x of global unknowns.

The conditions at the nodes require that there exist matrices V_ξ on the global unknowns, one for each node, with the following properties. At a node ξ we are given symmetric matrices H_ξ and K_ξ; symmetric matrices H'_ξ and K'_ξ with nonnegative entries, which bound errors; a real matrix Y'_ξ; and, for each global unknown j, numbers p_j, m_j and $\mathsf{w}_j > 0$, which depend on the node. For a real vector x let $|x|$ be the vector of the absolute values of its entries. For every x,

$$\begin{aligned} \langle \mathsf{H}_\xi x, x\rangle &\le \langle \mathsf{N}_\xi x, x\rangle + \sum_{\xi'} \langle \mathsf{V}_{\xi'} x, x\rangle + \langle \mathsf{H}'_\xi\, |x|, |x|\rangle, \\ \langle \mathsf{V}_\xi x, x\rangle &\le \langle \mathsf{K}_\xi x, x\rangle - \sum_{j \notin \mathcal{U}_\xi} \mathsf{m}_j\, x_j^2 + \langle \mathsf{K}'_\xi\, |x|, |x|\rangle, \end{aligned} \tag{B.39}$$

where ξ' runs through the children of ξ; for every unknown i,

$$\mathsf{w}_i \sum_j \frac{1}{\mathsf{w}_j}\Big(\big|(\mathsf{H}_\xi - \operatorname{diag}_{\mathcal{U}_\xi}(\mathsf{p}) - \mathsf{Y}'_\xi \mathsf{Y}'^{\mathsf{T}}_\xi + \mathsf{Y}_\xi \mathsf{Y}^{\mathsf{T}}_\xi - \mathsf{K}_\xi)_{ij}\big| + (\mathsf{H}'_\xi)_{ij} + (\mathsf{K}'_\xi)_{ij}\Big) \le \mathsf{m}_i, \tag{B.40}$$

where $\operatorname{diag}_{\mathcal{U}_\xi}(\mathsf{p})$ is the diagonal matrix with the entries p_j for $j \in \mathcal{U}_\xi$ and 0 for the other unknowns; and $\underline{\mathsf{p}}_{\xi,j} \le \mathsf{p}_j - \mathsf{m}_j$ for $j \in \mathcal{U}_\xi$.

At a node the conditions may instead be imposed in complex coordinates $\mathsf{z} = \mathsf{X}_\xi x$, each of which is the sum of an unknown x_j and of i times a real multiple of an unknown $x_{j'}$, where either term may be absent. The matrices H_ξ, K_ξ, H'_ξ and K'_ξ then act on the coordinates, with H_ξ and K_ξ Hermitian, the numbers p, m and w belong to the coordinates, and some of the coordinates are pivots. With a number $t_\xi \ge 0$ and with the vector $|\mathsf{z}|$ of the moduli of the coordinates, the conditions (B.39) become

$$t_\xi \langle \mathsf{H}_\xi \mathsf{z}, \mathsf{z}\rangle \le \langle \mathsf{N}_\xi x, x\rangle + \sum_{\xi'} \langle \mathsf{V}_{\xi'} x, x\rangle + t_\xi \langle \mathsf{H}'_\xi\, |\mathsf{z}|, |\mathsf{z}|\rangle,$$

$$\langle \mathsf{V}_\xi x, x\rangle \le t_\xi \Big(\langle \mathsf{K}_\xi \mathsf{z}, \mathsf{z}\rangle - \sum_l \mathsf{m}_l\, |\mathsf{z}_l|^2 + \langle \mathsf{K}'_\xi\, |\mathsf{z}|, |\mathsf{z}|\rangle\Big),$$

where the sum is over the coordinates that are not pivots. In (B.40) the pivot coordinates take the place of $\mathcal{U}_\xi$, the matrix Y'_ξ and a matrix in the place of Y_ξ are complex, and the transposes and the absolute values become adjoints and moduli. In a pivot coordinate the multiple is ± 1, and $j \ne j'$ if both terms are present; the set $\mathcal{U}_\xi$ consists of the unknowns that occur in the pivot coordinates, with $\underline{\mathsf{p}}_{\xi,j} \le t_\xi \sum_l (\mathsf{p}_l - \mathsf{m}_l)$, the sum being over the pivot coordinates z_l in which x_j occurs; and the real and the imaginary parts of the vectors $\sqrt{t_\xi}\mathsf{X}^*_\xi \mathsf{y}$, where y runs through the columns of the complex matrix, are distinct columns of Y_ξ. Three further conditions concern the whole tree: the root matrix V_ξ is nonnegative; every global unknown, determined or free, is a pivot at some node; and the total number of columns of the matrices Y_ξ is at most the number of free unknowns.

At a real node (B.40) charges the weighted row discrepancies and the entrywise errors $\mathsf{H}'_\xi, \mathsf{K}'_\xi$ to m_i. For a pivot, at most $\mathsf{p}_i - \mathsf{m}_i$ remains as a lower coefficient. These row and pivot conditions give a lower form bound for every coordinate vector in Lemma B.7. Positivity of the summed pivot bounds and the column count are separate tree conditions.

Lemma B.7. *Suppose that the conditions on the tree hold. Then, for every real vector x of global unknowns,*

$$\langle \mathsf{N}\, x, x\rangle \ge \sum_\xi \Big(\sum_{j \in \mathcal{U}_\xi} \underline{\mathsf{p}}_{\xi,j}\, x_j^2 - |\mathsf{Y}^{\mathsf{T}}_\xi x|^2\Big). \tag{B.41}$$

In particular $\langle \mathsf{N}\, x, x\rangle > 0$ for every $x \ne 0$ that is orthogonal to the columns of all the matrices Y_ξ.

Proof. We fix a node ξ and omit the index ξ of its data. Let B be the symmetric matrix whose entries stand between the bars in (B.40), so that $\langle \mathsf{H}x, x\rangle = \langle Bx, x\rangle + \sum_{j\in\mathcal{U}} \mathsf{p}_j\, x_j^2 + |\mathsf{Y}'^{\mathsf{T}} x|^2 - |\mathsf{Y}^{\mathsf{T}} x|^2 + \langle \mathsf{K}x, x\rangle$. Since $2\,|x_i|\,|x_j| \le \frac{\mathsf{w}_i}{\mathsf{w}_j}\, x_i^2 + \frac{\mathsf{w}_j}{\mathsf{w}_i}\, x_j^2$ and the matrices B, H' and K' are symmetric, (B.40) gives

$$-\langle Bx, x\rangle + \langle (\mathsf{H}' + \mathsf{K}')|x|, |x|\rangle \le \sum_{i,j} (|B_{ij}| + \mathsf{H}'_{ij} + \mathsf{K}'_{ij}) \frac{\mathsf{w}_i}{\mathsf{w}_j}\, x_i^2 \le \sum_i \mathsf{m}_i\, x_i^2.$$

Insert this error bound into the first line of (B.39), then use the second line to replace the contribution of K_ξ. Discard the nonnegative square $|\mathsf{Y}'^{\mathsf{T}} x|^2$ and use $\underline{\mathsf{p}}_{\xi,j} \le \mathsf{p}_j - \mathsf{m}_j$ on the

pivots. The result is

$$\langle \mathsf{N}_\xi x, x\rangle + \sum_{\xi'} \langle \mathsf{V}_{\xi'} x, x\rangle \ge \langle \mathsf{V}_\xi x, x\rangle + \sum_{j\in\mathcal{U}_\xi} \underline{\mathsf{p}}_{\xi,j}\, x_j^2 - |\mathsf{Y}_\xi^{\mathsf{T}} x|^2. \tag{B.42}$$

At a node with complex coordinates the same computation for the Hermitian forms at $\mathsf{z} = \mathsf{X}_\xi x$, multiplied by t_ξ, gives (B.42) again: for a pivot coordinate, $|\mathsf{z}_l|^2$ is the sum of the squares of the unknowns that occur in it, so that the sum of $t_\xi\,(\mathsf{p}_l - \mathsf{m}_l)\,|\mathsf{z}_l|^2$ over the pivot coordinates is at least $\sum_{j\in\mathcal{U}_\xi} \underline{\mathsf{p}}_{\xi,j}\, x_j^2$; and $t_\xi\,|\mathsf{y}^*\mathsf{z}|^2$ is the sum of the squares of the products of x with the real and the imaginary part of $\sqrt{t_\xi}\mathsf{X}_\xi^*\mathsf{y}$, which are columns of Y_ξ. We add (B.42) over the nodes. Every node other than the root is a child of exactly one node, so that its matrix V_ξ occurs once on each side, and that of the root, which occurs on the right only, is nonnegative. This gives (B.41). If x is orthogonal to all the columns, the right-hand side of (B.41) is $\sum_j (\sum_\xi \underline{\mathsf{p}}_{\xi,j}) x_j^2$, where the inner sum is over the nodes of which j is a pivot; it is positive for $x \ne 0$. □

The last condition is a bound for A_0: for every real vector x_{g} of global unknowns, with its entries β and ζ and with the number n_b of (B.34),

$$\langle \mathsf{A}_0\, x_{\mathrm{g}}, x_{\mathrm{g}}\rangle \le -\Big(\frac{3\,\omega^2}{50\cdot 4\pi} + \mathsf{n}_b\Big)|\beta|^2 - \frac{|\zeta|^2}{4\pi\cdot 1000}. \tag{B.43}$$

The number n_b is bounded from below: it is at least a sum over the cells of given nonnegative numbers and of the integrals over the cell, with given positive weights, of the squares of the errors made when ∇b_i is replaced on some harmonics by simpler functions.

Proposition B.8. *Suppose that the data satisfy the conditions of this appendix. Then, for every $\alpha \in \mathcal{K}$ and all v, β and F as in (7.18), we have $\mathcal{I}^- < \infty$, and (B.21) holds.*

Proof. By Proposition B.4, $\mathcal{I}^- < \infty$. If $\mathcal{I}^+ = \infty$, the left-hand side of (B.21) is $+\infty$, and there is nothing to prove. Let $\mathcal{I}^+ < \infty$; then all the integrals below converge absolutely, by Lemma B.3. Let x_{p} be as in Proposition B.4, let x_{f} be arbitrary, and let $x_{\mathrm{g}} = (x_{\mathrm{p}}, x_{\mathrm{f}})$. In a cell, $\mathcal{I}_\iota^\circ \ge 0$ almost everywhere for every harmonic that is not resolved, by Lemma B.5, since the regions of the cell and the harmonic cover the cell; hence the sum on the left of (B.33) is nonnegative. On the complement of the cells let $\mathcal{L}_\iota = \mathcal{L}(\nabla\hat v_\iota, \hat v_\iota)$. Then $\mathcal{L}_\iota \ge 0$ and $\mathcal{I}_\iota - \mathcal{L}_\iota \ge \mathcal{I}_\iota^\circ$, because s_1, s_2 and s_3 are nonnegative, $\mathsf{c}^* \ge \sqrt{7/2}$, and the functions a_1', a_1 and a_0 are nonnegative there. By Lemmas B.5 and B.6, and by the last assertion of Proposition B.4 for the regions $\{r > a \text{ or } z > a\}$, the integral of $\mathcal{I}_\iota^\circ$ over each region given for ι is nonnegative, and so are its integral over the complement, in which these regions lie and which they cover without overlap, and that of $\mathcal{I}_\iota - \mathcal{L}_\iota$. The sum over ι of the integrals of the functions $|\mathcal{I}_\iota|$ is finite; hence so is that of the functions $\mathcal{L}_\iota$, and by (B.32) the integral of $\sum_\iota \mathcal{I}_\iota + \bar J$ over the complement is at least the sum over ι of the integrals of $\mathcal{I}_\iota - \mathcal{L}_\iota$, which is nonnegative. We now add the inequalities (B.33) of the cells, the inequality (B.34), the integral over the complement, and (B.43). The numbers $\mathsf{r}_{\mathsf{Q},0}, \dots, \mathsf{r}_{\mathsf{Q},3}$ and the terms with n_b cancel, and by (B.38) and (B.23) we obtain

$$\begin{aligned} 4\pi\,\langle \mathsf{N}\, x_{\mathrm{g}}, x_{\mathrm{g}}\rangle \le{}& \int_0^1 \int_{\mathbb{R}^3} \Big(\sum_\iota \mathcal{I}_\iota + J\Big)\, dy\, d\theta - \tfrac{3}{50}\,\omega^2|\beta|^2 + \sum_i \beta_i\,\langle b_i, \mathsf{M} v\rangle_{\mathcal{H}} + \tfrac{3}{50}\,[F]^2 \\ &+ 2\,\zeta\cdot\ell(v) - \frac{|\zeta|^2}{1000}. \end{aligned} \tag{B.44}$$

Completing the square gives $2\,\zeta\cdot\ell(v) - |\zeta|^2/1000 = 1000\,|\ell(v)|^2 - |\zeta - 1000\ell(v)|^2/1000 \le 1000\,|\ell(v)|^2$. Hence the right-hand side of (B.44) is at most the left-hand side of (B.21) minus its right-hand side, for every x_{f}. It remains to find one x_{f} for which $\langle \mathsf{N}\, x_{\mathrm{g}}, x_{\mathrm{g}}\rangle \ge 0$.

We need to choose x_{f} so that $(x_{\mathrm{p}}, x_{\mathrm{f}})$ is orthogonal to every column of every Y_ξ while x_{p} stays fixed. Write $\mathcal{Y}^\perp$ for this orthogonal subspace. The zero input tests whether a nonzero vector in $\mathcal{Y}^\perp$ can have every prescribed coordinate equal to zero. With $v = 0$, $\beta = 0$ and $F = 0$, we have $x_{\mathrm{p}} = 0$, and the right side of (B.44) becomes $-|\zeta|^2/1000$. Thus $\langle \mathsf{N}x, x\rangle \le 0$ on the free-coordinate space $E_{\mathrm{f}} = \{(0, x_{\mathrm{f}})\}$. Lemma B.7 makes the same form strictly positive on nonzero vectors in $\mathcal{Y}^\perp$. The two subspaces therefore intersect only at zero, so the projection $(x_{\mathrm{p}}, x_{\mathrm{f}}) \mapsto x_{\mathrm{p}}$ is injective on $\mathcal{Y}^\perp$. In the full space E the column count gives $\operatorname{codim}_E \mathcal{Y}^\perp \le \dim E_{\mathrm{f}}$. Thus $\dim \mathcal{Y}^\perp$ is at least the dimension of the prescribed-coordinate space. The injective projection is consequently onto, or equivalently $E = \mathcal{Y}^\perp \oplus E_{\mathrm{f}}$. Returning to v, β and F, we write $(x_{\mathrm{p}}, 0)$ as the sum of an element of $\mathcal{Y}^\perp$ and of a vector $(0, -x_{\mathrm{f}})$. Then $(x_{\mathrm{p}}, x_{\mathrm{f}}) \in \mathcal{Y}^\perp$, and $\langle \mathsf{N}\, x_{\mathrm{g}}, x_{\mathrm{g}}\rangle \ge 0$ for this x_{f}. □

B.4. The other conditions and the statement. The other estimates that a certificate assumes for every perturbation follow from bounds on finitely many norms and pairings of the given fields. We first record the inequalities used in these reductions: the Sobolev inequality (3.1), with its explicit constant,

$$\|u\|_{L^6} \le \tfrac{2}{\sqrt{3}} \|\nabla u\|_{L^2} \qquad \text{for } u \in X, \tag{B.45}$$

which for scalar functions is Galdi [56, (II.3.11), p. 56]. Applying it componentwise and using $\|u\|_{L^6}^2 = \||u|^2\|_{L^3} \le \sum_i \|u_i\|_{L^6}^2$ gives the same constant for a vector field. For $u \in X$ with $\Delta u \in L^2$, the lower bound $w \ge w_* > 0$ and (7.48) also give

$$\begin{aligned} \sqrt{w_*}\, \|\nabla u\|_{L^2} \le |u|_w, \qquad \|\Delta u\|_{L^2} \le |u|_w, \\ 2\sqrt{w_*}\, \|\nabla u\|_{L^2}\, \|\Delta u\|_{L^2} \le |u|_w^2, \qquad \|u\|_{L^\infty} \le c_w\, |u|_w. \end{aligned} \tag{B.46}$$

The first two bounds follow by retaining one term in (7.14); the third is $2ab \le a^2 + b^2$ with $a = \|\Delta u\|_{L^2}$ and $b = \sqrt{w_*}\|\nabla u\|_{L^2}$. Appendix B uses $w \equiv w_* = \frac{39}{50}$.

Lemma B.9. *Let W_0^{a} be the approximate profile for $\alpha = 0$, which is the field W_0^0 of (B.9). Let n_0^∞, $\mathsf{n}_0^{(2)}$, n_1^∞, $\mathsf{n}_1^{(3)}$, $\mathsf{n}_{\mathrm{r}}^{(3)}$ and $\mathsf{s}_- \ge 0$ be numbers such that, for every $\alpha \in \mathcal{K}$,*

$$\begin{aligned} \mathsf{n}_0^\infty &\ge \sup |W_0^{\mathrm{a}} - W^{\mathrm{r}}|, & \mathsf{n}_0^{(2)} &\ge \sup_\theta \big\| |\nabla (W_0^{\mathrm{a}} - W^{\mathrm{r}})(\theta)|_{\mathrm{op}} \big\|_{L^2}, \\ \mathsf{n}_1^\infty &\ge \sup |W_\alpha^{\mathrm{a}} - W_0^{\mathrm{a}}|, & \mathsf{n}_1^{(3)} &\ge \sup_\theta \big\| |\nabla (W_\alpha^{\mathrm{a}} - W_0^{\mathrm{a}})(\theta)|_{\mathrm{op}} \big\|_{L^3}, \\ \mathsf{n}_{\mathrm{r}}^{(3)} &\ge \sup_\theta \big\| |\nabla W^{\mathrm{r}}(\theta)|_{\mathrm{op}} \big\|_{L^3}, & \mathbb{S}(\theta, y) &\ge -\mathsf{s}_-\, I \quad \textit{for all } (\theta, y), \end{aligned} \tag{B.47}$$

where the suprema without an index are over $\mathbb{R}\times\mathbb{R}^3$ and the last inequality is between quadratic forms, and let c_{r}, c_3' and c_3'' be numbers such that

$$\begin{aligned} c_w\, \mathsf{n}_0^{(2)} + \frac{\mathsf{n}_0^\infty}{\sqrt{w_*}} + \frac{2}{\sqrt{3}} \frac{\mathsf{n}_1^{(3)}}{\sqrt{w_*}} + \frac{\mathsf{n}_1^\infty}{\sqrt{w_*}} \le c_{\mathrm{r}}, \qquad \frac{\mathsf{s}_-}{w_*} + \frac{2}{\sqrt{3}} \frac{\mathsf{n}_{\mathrm{r}}^{(3)}}{2\sqrt{w_*}} \le c_3', \\ c_w\, \mathsf{n}_0^{(2)} + \frac{\mathsf{n}_0^\infty}{2\sqrt{w_*}} + \frac{2}{\sqrt{3}} \frac{\mathsf{n}_1^{(3)}}{2\sqrt{w_*}} + \frac{\mathsf{n}_1^\infty}{2\sqrt{w_*}} \le c_3''. \end{aligned} \tag{B.48}$$

Then, for every $\alpha \in \mathcal{K}$, the bound (7.19) holds for every $v \in \mathcal{D}_{S(\alpha)}$, and the lower bound (7.22) holds, for every $\theta \in \mathbb{R}$ and every $u \in X$ with $\Delta u \in L^2$, with every number $c_3 \ge c_3' + c_3''$.

Proof. Let $\alpha \in \mathcal{K}$, and let $Z_0 = W_0^{\mathrm{a}} - W^{\mathrm{r}}$ and $Z_1 = W_\alpha^{\mathrm{a}} - W_0^{\mathrm{a}}$, so that $W_\alpha^{\mathrm{a}} - W^{\mathrm{r}} = Z_0 + Z_1$. Let $\theta \in \mathbb{R}$ and $u \in X$ with $\Delta u \in L^2$; all fields are taken at the time θ. For a field a of class

C^1 we have $|(a\cdot\nabla)u|\le|a|\,|\nabla u|$ and $|(u\cdot\nabla)a|\le|\nabla a|_{\mathrm{op}}\,|u|$ at every point. Hence, by (B.47), by Hölder's inequality with (B.45), and by the last inequality of (B.46),

$$\begin{aligned}\|((Z_0+Z_1)\cdot\nabla)u\|_{L^2}&\le(\mathsf{n}_0^{\infty}+\mathsf{n}_1^{\infty})\,\|\nabla u\|_{L^2},\\ \|(u\cdot\nabla)Z_0\|_{L^2}&\le\mathsf{n}_0^{(2)}\,\|u\|_{L^\infty}\le c_w\,\mathsf{n}_0^{(2)}\,|u|_w,\\ \|(u\cdot\nabla)Z_1\|_{L^2}&\le\mathsf{n}_1^{(3)}\,\|u\|_{L^6}\le\tfrac{2}{\sqrt3}\,\mathsf{n}_1^{(3)}\,\|\nabla u\|_{L^2}.\end{aligned}\tag{B.49}$$

Replace $\|\nabla u\|_{L^2}$ in (B.49) by $|u|_w/\sqrt{w_*}$. The first inequality of (B.48) then bounds the sum by $c_{\mathrm r}|u|_w$. For $v\in\mathcal{D}_{S(\alpha)}$, this holds with $u=v(\theta)$ for almost every θ. The field in (7.19) is the sum $((Z_0+Z_1)\cdot\nabla)v+(v\cdot\nabla)Z_0+(v\cdot\nabla)Z_1$. Squaring its bound and integrating over one period gives membership in $\mathcal{H}$ and (7.19).

For (7.22), first use the reference profile $W=W^{\mathrm r}(\theta)$. It is divergence-free, of class C^1, and bounded together with its gradient (Subsection B.1). By (3.2) and Plancherel's identity, $\Delta u\in L^2$ gives square-integrable second derivatives. For each component set $g=\nabla u_i$. Then $g, W\cdot g\in H^1(\mathbb{R}^3)$ and $\partial_k g_j=\partial_j g_k$, so integration by parts gives

$$-\int_{\mathbb{R}^3}(W\cdot\nabla)u_i\,\Delta u_i\,dy=\int_{\mathbb{R}^3}\nabla(W\cdot g)\cdot g\,dy=\int_{\mathbb{R}^3}\sum_{j,k}\partial_k W_j\,g_j\,g_k\,dy+\frac12\int_{\mathbb{R}^3}W\cdot\nabla|g|^2\,dy.$$

The last integral vanishes, because $\operatorname{div}W=0$, W is bounded and $|g|^2\in W^{1,1}(\mathbb{R}^3)$, and the integrand of the first is $g\cdot\mathbb{S}(\theta)\,g\ge-\mathsf{s}_-\,|g|^2$. One may first insert a cut-off equal to one on B_R and zero outside B_{2R}, with gradient bounded by C/R. The resulting boundary term is bounded by $C\|W\|_{L^\infty}R^{-1}\int_{R<|y|<2R}|g|^2\,dy$ and tends to zero. We sum over i and bound the other term of the linearization by Hölder's inequality with (B.45):

$$-\int_{\mathbb{R}^3}(W\cdot\nabla)u\cdot\Delta u\,dy\ge-\mathsf{s}_-\,\|\nabla u\|_{L^2}^2,\qquad\left|\int_{\mathbb{R}^3}(u\cdot\nabla)W\cdot\Delta u\,dy\right|\le\tfrac{2}{\sqrt3}\,\mathsf{n}_{\mathrm r}^{(3)}\,\|\nabla u\|_{L^2}\,\|\Delta u\|_{L^2}.$$

Since $\int(|\Delta u|^2+\frac14|\nabla u|^2)\,dy\ge0$, the first and the third inequality of (B.46) and the second of (B.48) give that the left-hand side of (7.22) with $W^{\mathrm r}$ in place of $W^{\mathrm a}_\alpha$ is at least $-c_3'\,|u|_w^2$. The difference of the two left-hand sides is $-\int[((Z_0+Z_1)\cdot\nabla)u+(u\cdot\nabla)Z_0+(u\cdot\nabla)Z_1]\cdot\Delta u\,dy$. By (B.49) and the Cauchy–Schwarz inequality its absolute value is at most

$$\Big(\mathsf{n}_0^{\infty}+\mathsf{n}_1^{\infty}+\tfrac{2}{\sqrt3}\,\mathsf{n}_1^{(3)}\Big)\|\nabla u\|_{L^2}\,\|\Delta u\|_{L^2}+c_w\,\mathsf{n}_0^{(2)}\,|u|_w\,\|\Delta u\|_{L^2}\le c_3''\,|u|_w^2,$$

by the second and the third inequality of (B.46) and the third of (B.48). Thus (7.22) holds with $c_3=c_3'+c_3''$, and hence with every larger number. □

For each $j\in\mathcal{J}$, we now bound the quadratic pairing (7.32), the linear pairing (7.33), and the residual pairing (7.34), in that order. The first uses a bound on either $\|\psi_j(\theta)\|_{L^3}$ or $\|\psi_j(\theta)\|_{L^2}$, uniform in θ.

Lemma B.10. *Let $j\in\mathcal{J}$ and $\mathsf{n}_\psi\ge0$. Suppose that $\mathsf{n}_\psi\ge\sup_\theta\|\psi_j(\theta)\|_{L^3}$ and $\frac{2}{\sqrt3\,w_*}\,\mathsf{n}_\psi\le\tilde a_j$, or that $\mathsf{n}_\psi\ge\sup_\theta\|\psi_j(\theta)\|_{L^2}$ and $\frac{c_w}{\sqrt{w_*}}\,\mathsf{n}_\psi\le\tilde a_j$. Then (7.32) holds for every $\alpha\in\mathcal{K}$ and every $v\in\mathcal{D}_{S(\alpha)}$.*

Proof. For almost every θ, apply Hölder's inequality to the three factors v, ∇v and ψ_j. The exponents $(6,2,3)$ in the first case give the bound $\|v(\theta)\|_{L^6}\|\nabla v(\theta)\|_{L^2}\mathsf{n}_\psi$ for $\int_{\mathbb{R}^3}|(v\cdot\nabla)v\cdot\psi_j|\,dy$; the exponents $(\infty,2,2)$ in the second give $\|v(\theta)\|_{L^\infty}\|\nabla v(\theta)\|_{L^2}\mathsf{n}_\psi$. By (B.45) and (B.46) this is at most $\frac{2}{\sqrt3\,w_*}\,\mathsf{n}_\psi\,|v(\theta)|_w^2$ or $\frac{c_w}{\sqrt{w_*}}\,\mathsf{n}_\psi\,|v(\theta)|_w^2$ respectively, hence at most $\tilde a_j\,|v(\theta)|_w^2$, and an integration over $(0,1)$ gives (7.32). □

The linear estimate (7.33) requires bounds on several fields paired with the perturbation. Throughout the rest of this subsection, $\iint$ means integration over $(0,1)\times\mathbb{R}^3$ with measure $dy\,d\theta$. We say that a measurable field R on $\mathbb{R}\times\mathbb{R}^3$ of period 1 in θ has the *dual bound* n if $\mathsf{R}=\mathsf{R}'+\mathsf{R}''$ and there are numbers $\mathsf{n}',\mathsf{n}''\ge 0$ with $\mathsf{n}'+\mathsf{n}''\le\mathsf{n}$ such that $2\,\||y|\,\mathsf{R}'\|_{\mathcal{H}}\le\mathsf{n}'$ and R'' is of the following kind: each of the components $\mathsf{R}''_1+i\,\mathsf{R}''_2$, $\mathsf{R}''_1-i\,\mathsf{R}''_2$ and R''_3 is a finite sum of terms $e^{2\pi ik'\theta}g(y)$, where k' is an integer and g is a measurable function with an azimuthal wave number k, that is, $g(R_{\phi_0}y)=e^{ik\phi_0}g(y)$ for every angle ϕ_0, with distinct pairs (k',k) in each component, and the sum of the numbers $(|k|+\frac12)^{-2}\int_{\mathbb{R}^3}|y|^2\,|g|^2\,dy$ over all the terms, those of the first two components counted with the factor $\frac12$, is at most n''^2.

Lemma B.11. *Let R have the dual bound n, and let $v\in L^2_{\mathrm{loc}}(\mathbb{R};X)$ have period 1 in θ. Then $v\cdot\mathsf{R}$ is integrable on $(0,1)\times\mathbb{R}^3$, and $\bigl|\iint v\cdot\mathsf{R}\bigr|\le\mathsf{n}\,\|\nabla v\|_{\mathcal{H}}$.*

Proof. For $U\in C^\infty_c(\mathbb{R}^3;\mathbb{C})$ the inequality (B.37) with $a=0$, integrated over the rays from the origin, gives $\frac14\int_{\mathbb{R}^3}|U|^2\,|y|^{-2}\,dy\le\int_{\mathbb{R}^3}|\partial_\rho U|^2\,dy$, where ∂_ρ is the derivative along the rays. Since $|\partial_\rho U|\le|\nabla U|$, this gives Hardy's inequality $\int_{\mathbb{R}^3}|U|^2|y|^{-2}\,dy\le 4\int_{\mathbb{R}^3}|\nabla U|^2\,dy$; see Leray [70, (1.14), p. 204]. The density stated in Subsection 3.1, applied to the real and imaginary parts, extends both inequalities to $U\in L^6(\mathbb{R}^3;\mathbb{C})$ with $\nabla U\in L^2$. Let $\mathsf{R}=\mathsf{R}'+\mathsf{R}''$ as in the definition. The fields $|y|\,\mathsf{R}'$ and $|y|\,\mathsf{R}''$ belong to $\mathcal{H}$, so that $v\cdot\mathsf{R}$ is integrable and $|\iint v\cdot\mathsf{R}'|\le\||y|^{-1}v\|_{\mathcal{H}}\,\||y|\,\mathsf{R}'\|_{\mathcal{H}}\le\mathsf{n}'\,\|\nabla v\|_{\mathcal{H}}$. For the pairing with R'' we use that a function $U\in L^6(\mathbb{R}^3;\mathbb{C})$ with $\nabla U\in L^2$ and with an azimuthal wave number k satisfies

$$(|k|+\tfrac12)^2\int_{\mathbb{R}^3}\frac{|U|^2}{|y|^2}\,dy\le\int_{\mathbb{R}^3}|\nabla U|^2\,dy. \tag{B.50}$$

For $k=0$ this is Hardy's inequality. Let $k\neq0$, and let first U be smooth with compact support. The relation $U(R_{\phi_0}y)=e^{ik\phi_0}U(y)$ shows that U vanishes on the y_3 axis, and its derivative in ϕ_0 is $r\,\mathbf{e}_\phi\cdot\nabla U=ik\,U$. In the variables $\rho=|y|$, $\eta=y_3/|y|$ and ϕ we have $r^2=\rho^2(1-\eta^2)$, $dy=\rho^2\,d\rho\,d\eta\,d\phi$ and $|\nabla U|^2=|\partial_\rho U|^2+\rho^{-2}(1-\eta^2)\,|\partial_\eta U|^2+k^2\,r^{-2}\,|U|^2$. On each half circle on which ρ and ϕ are constant, an integration by parts, in which U vanishes at both ends, gives

$$\int_{-1}^1\Bigl((1-\eta^2)\,|\partial_\eta U|^2+\frac{k^2\,|U|^2}{1-\eta^2}\Bigr)\,d\eta-(k^2+|k|)\int_{-1}^1|U|^2\,d\eta=\int_{-1}^1(1-\eta^2)\left|\partial_\eta U+\frac{|k|\,\eta}{1-\eta^2}\,U\right|^2\,d\eta\ge0.$$

Integrated over ρ and ϕ, this is $\int_{\mathbb{R}^3}(|\nabla U|^2-|\partial_\rho U|^2)\,dy\ge(k^2+|k|)\int_{\mathbb{R}^3}|U|^2\,|y|^{-2}\,dy$, and with the inequality for $\partial_\rho U$ it gives (B.50), since $\frac14+k^2+|k|=(|k|+\frac12)^2$. A function U as in the statement is the limit, in the norm $\|\nabla\cdot\|_{L^2}$, of smooth functions U_n with compact support, and hence of their means $\frac{1}{2\pi}\int_0^{2\pi}e^{-ik\phi'}U_n(R_{\phi'}y)\,d\phi'$, which have the wave number k; by Hardy's inequality, (B.50) passes to the limit. Let now u be one of the functions v_1+iv_2, v_1-iv_2 and v_3, let R''_u be the corresponding component of R'', and for integers k' and k let

$$u_{k',k}(y)=\frac{1}{2\pi}\int_0^1\int_0^{2\pi}e^{-2\pi ik'\theta-ik\phi'}u(\theta,R_{\phi'}y)\,d\phi'\,d\theta,$$

which has the wave number k. The maps $u\mapsto e^{2\pi ik'\theta}u_{k',k}$ are orthogonal projections, with orthogonal ranges for distinct pairs (k',k), for the inner product $\iint\nabla u\cdot\overline{\nabla\tilde u}$, which the translations in θ and the rotations about the y_3 axis preserve; hence $\sum_{(k',k)}\|\nabla u_{k',k}\|^2_{L^2(\mathbb{R}^3)}\le\iint|\nabla u|^2$. The pairing of u with a term $e^{2\pi ik'\theta}\,g$ of R''_u is $\int_{\mathbb{R}^3}u_{k',k}\,\bar g\,dy$, by a rotation of the variable y, and by (B.50) it is at most $(|k|+\frac12)^{-1}\,\|\nabla u_{k',k}\|_{L^2(\mathbb{R}^3)}\,\||y|\,g\|_{L^2(\mathbb{R}^3)}$ in absolute value. Since the pairs (k',k) of the terms are distinct, the Cauchy–Schwarz inequality gives $|\iint u\,\overline{\mathsf{R}''_u}|^2\le(\mathsf{n}''_u)^2\iint|\nabla u|^2$, where $(\mathsf{n}''_u)^2$ is the sum of the numbers $(|k|+\frac12)^{-2}\int|y|^2\,|g|^2\,dy$ over the terms of

R''_u. Finally $v\cdot\mathsf{R}''$ is the sum of the three functions $u\,\overline{\mathsf{R}''_u}$, and $|\nabla v|^2$ that of the three functions $|\nabla u|^2$, in both cases with the factor $\frac12$ for the first two, and the Cauchy–Schwarz inequality gives $|\iint v\cdot\mathsf{R}''|\le \mathsf{n}''\,\|\nabla v\|_{\mathcal{H}}$. □

We fix $j\in\mathcal{J}$; the numbers and the fields G_j, U_j, F_j and q_j introduced in this paragraph depend on it. Since $h\ge3$ in every term of ψ_j, the fields ψ_j and $\partial_\theta\psi_j$ are bounded by a constant multiple of $(1+|y|)^{-3}$, and $\nabla\psi_j$ and $\nabla^2\psi_j$ by constant multiples of $(1+|y|)^{-4}$ and $(1+|y|)^{-5}$ (Subsection B.1). The constraints are first replaced by divergence-free fields: besides the fields Φ_l of Subsection B.1 we are given compactly supported fields $\tilde\Phi_1,\dots,\tilde\Phi_n$ of the form (B.6), each with a length, a radial phase and an integer in place of $k_{\rm rot}$ of its own, and numbers with $\|\sum_l Y_{jl}(\Phi_l-\tilde\Phi_l)\|_{\mathcal{H}}\le\mathsf{n}_\Phi$ and $(\sum_l Y_{jl}^2)^{1/2}\le\mathsf{n}_Y$. We are given two vector fields G_j and U_j, a field F_j of matrices and a scalar q_j, all of a form similar to (B.6) with piecewise polynomials of the Hermite type, which we do not write out. It gives that G_j and U_j are of class C^1, that at each time the Laplacians $\Delta\mathsf{G}_j$ and $\Delta\mathsf{U}_j$, the divergence $\operatorname{div}\mathsf{F}_j^{\mathsf{T}}$ of the transpose of F_j and the gradient ∇q_j, in the sense of distributions on $\mathbb{R}^3$, are functions, and that, for a constant C,

$$\begin{aligned}|\mathsf{G}_j|+|\nabla\mathsf{U}_j|+|\mathsf{F}_j|+|q_j|\le C\,(1+|y|)^{-2},\qquad |\nabla\mathsf{G}_j|\le C\,(1+|y|)^{-3},\\ |\Delta\mathsf{G}_j|+|\Delta\mathsf{U}_j|+|\operatorname{div}\mathsf{F}_j^{\mathsf{T}}|+|\nabla q_j|\le C\,|y|^{-1}(1+|y|)^{-2}.\end{aligned}\tag{B.51}$$

The scalar q_j is the auxiliary potential in the adjoint equation. Here the divergence of a field of matrices is that of Subsection 3.2. With the field W^0_α of (B.9) and the components $\psi_{j,k}$ of ψ_j, let

$$\begin{aligned}\mathsf{R}_{j,\alpha}=\ &-\frac{1}{S(0)}\,\partial_\theta\psi_j-\psi_j-\tfrac12(y\cdot\nabla)\psi_j-\Delta\psi_j-(W^0_\alpha\cdot\nabla)\psi_j+\sum_{k=1}^{3}\psi_{j,k}\,\nabla W^0_{\alpha,k}\\ &+\sum_{l=1}^{n}Y_{jl}\,\Delta\tilde\Phi_l+\nabla q_j+\Delta\mathsf{G}_j+w_*\,\Delta\mathsf{U}_j-\operatorname{div}\mathsf{F}_j^{\mathsf{T}}\end{aligned}\tag{B.52}$$

for $\alpha\in\mathcal{K}$. The first line is the formal adjoint of $v\mapsto\mathcal{T}_{S(0)}v+(W^0_\alpha\cdot\nabla)v+(v\cdot\nabla)W^0_\alpha$, applied to ψ_j. Thus $\mathsf{R}_{j,\alpha}$ is the residual of the adjoint equation with period $S(0)$, linearization about the base field W^0_α and right-hand side $-\sum_l Y_{jl}\Delta\tilde\Phi_l$, after adding the gradient and the three auxiliary-field terms in the second line. Integration by parts pairs those auxiliary fields with Δv and ∇v, as (B.56) records. Let $\mathsf{R}_{j,\alpha}$ have the dual bound n_R for every $\alpha\in\mathcal{K}$, let $\mathsf{n}_\mathsf{G}\ge\|\mathsf{G}_j\|_{\mathcal{H}}$, $\mathsf{n}_\mathsf{U}\ge\|\nabla\mathsf{U}_j\|_{\mathcal{H}}$ and $\mathsf{n}_\mathsf{F}\ge\|\mathsf{F}_j\|_{\mathcal{H}}$, let $\bar\varrho_i$ be numbers that bound $|\alpha_i|$ on $\mathcal{K}$ and $\varpi_i>0$ given numbers, and let $\mathsf{n}_{\rm v}\ge\|(\sum_i\varpi_i\,|W^i_\alpha|^2)^{1/2}\psi_j\|_{\mathcal{H}}$ for every $\alpha\in\mathcal{K}$ and $\mathsf{n}_{\rm d}\ge(\sum_i\varpi_i\,\mathsf{n}_i^2)^{1/2}$, where the field $\sum_k\psi_{j,k}\,\nabla W^i_{\alpha,k}$ has the dual bound n_i for every $\alpha\in\mathcal{K}$, and let $\partial_\theta\psi_j$ have the dual bound n_θ.

Lemma B.12. *Let $j\in\mathcal{J}$, and let the fields and the numbers above be given. Then, for every $\alpha\in\mathcal{K}$, every $v\in\mathcal{D}_{S(\alpha)}$ and every $F\in\mathcal{H}$ with $F\simeq\mathcal{T}_{S(\alpha)}v+\mathcal{C}_\alpha v$, the bound (7.33) holds with every number a_j such that*

$$\begin{aligned}&\Big(\mathsf{n}_\mathsf{G}^2+\frac{(w_*\,\mathsf{n}_\mathsf{U}+\mathsf{n}_\mathsf{F}+\mathsf{n}_\mathsf{R})^2}{w_*}\Big)^{1/2}+\frac{1}{\sqrt{w_*}}\Big[\Big(\sum_i\frac{\bar\varrho_i^2}{\varpi_i}\Big)^{1/2}(\mathsf{n}_{\rm v}+\mathsf{n}_{\rm d})+\Big(\sum_i|\varsigma_i|\,\bar\varrho_i\Big)\mathsf{n}_\theta\Big]\\ &\qquad+\mathsf{n}_\Phi+\frac{\mathsf{n}_Y\,c_\Xi}{\sqrt{w_*}}\le a_j.\end{aligned}\tag{B.53}$$

Proof. Let $\alpha\in\mathcal{K}$ and $S=S(\alpha)$, let v and F be as in the statement, and write ψ for ψ_j and R for $\mathsf{R}_{j,\alpha}$. The field ψ is divergence-free and belongs to $\mathcal{H}$. The fields $\mathcal{T}_Sv$ and $\mathcal{C}_\alpha v$ belong to $\mathcal{H}$ (Lemmas 7.4 and 7.7), and for fields of $\mathcal{H}$ the relation $\simeq$ is the equality of the Leray projections

(Subsection 7.1). Hence $\langle F,\psi\rangle_{\mathcal{H}} = \langle \mathcal{T}_S v,\psi\rangle_{\mathcal{H}} + \langle \mathcal{C}_\alpha v,\psi\rangle_{\mathcal{H}}$, and we move the operators to ψ. First,

$$\langle \mathcal{T}_S v,\psi\rangle_{\mathcal{H}} = \iint \Big[v\cdot\Big(-\frac{1}{S}\,\partial_\theta\psi - \psi - \tfrac{1}{2}(y\cdot\nabla)\psi\Big) + \nabla v\cdot\nabla\psi\Big], \tag{B.54}$$

where $\operatorname{div} y = 3$ gives the transpose of $\frac{1}{2}+\frac{1}{2}y\cdot\nabla$ as $\frac{1}{2}-\frac{3}{2}-\frac{1}{2}y\cdot\nabla = -1-\frac{1}{2}y\cdot\nabla$. This is the definition of $\mathcal{T}_S v$ in the sense of distributions, applied to ψ times a cut-off in y, which is then removed: by the decay of ψ and (B.45), the functions $|v|\,(|\psi|+|\partial_\theta\psi|+|y|\,|\nabla\psi|)$ and $|\nabla v|\,(|\psi|+|\nabla\psi|)$ are integrable on $(0,1)\times\mathbb{R}^3$. Secondly, for almost every θ and every field $A\in L^2(\mathbb{R}^3)$ of matrices with $\operatorname{div} A^{\mathsf{T}}\in L^{6/5}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} v(\theta)\cdot\operatorname{div} A^{\mathsf{T}}\,dy = -\int_{\mathbb{R}^3}\nabla v(\theta)\cdot A\,dy, \tag{B.55}$$

by the density of $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ in X for the norm $\|\nabla u\|_{L^2}$ (Subsection 3.1) and by (3.1). By the decay of ψ and (B.51), (B.55) applies to $\nabla\psi$, $\nabla\tilde\Phi_l$, $\nabla\mathsf{G}_j$, $\nabla\mathsf{U}_j$ and F_j; to q_j times the identity matrix, for which its right-hand side vanishes; and to the matrix with the entries $W^0_{\alpha,i}\,\psi_k$, for which it reads $\int v\cdot(W^0_\alpha\cdot\nabla)\psi\,dy = -\int(W^0_\alpha\cdot\nabla)v\cdot\psi\,dy$. Moreover $\int\nabla v(\theta)\cdot\nabla G\,dy = -\int\Delta v(\theta)\cdot G\,dy$ for every $G\in H^1(\mathbb{R}^3;\mathbb{R}^3)$; we use this for G_j and for $\Phi_l-\tilde\Phi_l$.

We write $\mathcal{C}_\alpha v$ with $W^{\mathrm{a}}_\alpha = W^0_\alpha + \sum_i \alpha_i W^i_\alpha$, and we use that $(v\cdot\nabla)W\cdot\psi = v\cdot\sum_k\psi_k\nabla W_k$ for a field W of class C^1 and that $\frac{1}{S} = \frac{1}{S(0)}+\varsigma\cdot\alpha$. With (B.54) and (B.55) this expresses $\langle F,\psi\rangle_{\mathcal{H}}$ as the pairing of v with the first line of (B.52) plus the second line of (B.56) below. Substituting for the first line of (B.52), applying (B.55) to the other matrices, and subtracting $(Y(\alpha)\,\ell(v))_j = \sum_l Y_{jl}\,\langle\nabla v,\Xi_l\rangle_{\mathcal{H}}$, we obtain

(B.56)

$$\begin{aligned}\langle F,\psi\rangle_{\mathcal{H}} - \big(Y(\alpha)\ell(v)\big)_j = {}& \iint v\cdot\mathsf{R} - \langle\Delta v,\mathsf{G}_j\rangle_{\mathcal{H}} + \langle\nabla v, w_*\,\nabla\mathsf{U}_j - \mathsf{F}_j\rangle_{\mathcal{H}}\\ &+\sum_i\alpha_i\Big[\langle(W^i_\alpha\cdot\nabla)v,\psi\rangle_{\mathcal{H}} + \iint v\cdot\sum_k\psi_k\,\nabla W^i_{\alpha,k}\Big] - (\varsigma\cdot\alpha)\iint v\cdot\partial_\theta\psi\\ &+\Big\langle\Delta v,\sum_l Y_{jl}\,(\Phi_l-\tilde\Phi_l)\Big\rangle_{\mathcal{H}} - \Big\langle\nabla v,\sum_l Y_{jl}\,(\Xi_l-\nabla\Phi_l)\Big\rangle_{\mathcal{H}}.\end{aligned}$$

We estimate the three lines of (B.56) in order: the potential and dual-bound terms, the parameter and period changes, and the constraint replacement. They give the three contributions in (B.53), each multiplied by $\|v\|_w$. By Lemma B.11, the first line is at most $\mathsf{n}_{\mathsf{G}}\,\|\Delta v\|_{\mathcal{H}} + (w_*\,\mathsf{n}_{\mathsf{U}} + \mathsf{n}_{\mathsf{F}} + \mathsf{n}_{\mathsf{R}})\,\|\nabla v\|_{\mathcal{H}}$ in absolute value, which is at most the first term of (B.53) times $(\|\Delta v\|^2_{\mathcal{H}} + w_*\,\|\nabla v\|^2_{\mathcal{H}})^{1/2} \le \|v\|_w$. In the second line, $|\sum_i\alpha_i W^i_\alpha| \le (\sum_i\bar\varrho_i^2/\varpi_i)^{1/2}(\sum_i\varpi_i\,|W^i_\alpha|^2)^{1/2}$, so that the sum of the first pairings is at most $(\sum_i\bar\varrho_i^2/\varpi_i)^{1/2}\,\mathsf{n}_{\mathrm{v}}\,\|\nabla v\|_{\mathcal{H}}$; by Lemma B.11 the sum of the second ones is at most $\sum_i\bar\varrho_i\,\mathsf{n}_i\,\|\nabla v\|_{\mathcal{H}} \le (\sum_i\bar\varrho_i^2/\varpi_i)^{1/2}\,\mathsf{n}_{\mathrm{d}}\,\|\nabla v\|_{\mathcal{H}}$, and the last term is at most $(\sum_i|\varsigma_i|\,\bar\varrho_i)\,\mathsf{n}_\theta\,\|\nabla v\|_{\mathcal{H}}$. The third line is at most $\mathsf{n}_\Phi\,\|\Delta v\|_{\mathcal{H}} + \mathsf{n}_Y\,c_\Xi\,\|\nabla v\|_{\mathcal{H}}$, by the bound for the fields $\Xi_i-\nabla\Phi_i$ of Subsection B.1. Since $\|\Delta v\|_{\mathcal{H}}\le\|v\|_w$ and $\sqrt{w_*}\,\|\nabla v\|_{\mathcal{H}}\le\|v\|_w$, this gives (7.33). □

The number o_j is obtained from an expansion in the ball $B_{r_\chi} = \{|y|<r_\chi\}$. For $0\le i\le n$ let W^i_0 be the field W^i_α of (B.9) for $\alpha=0$, and let the function q^i of Subsection B.1 be taken

with the radial phase f. With the second derivatives of Subsection B.1, let

(B.57)
$$\begin{aligned}
F^{(0)} &= \tfrac{1}{S(0)}\,\partial_\theta W_0^0 + \tfrac12 W_0^0 + \tfrac12 (y\cdot\nabla) W_0^0 - \Delta W_0^0 + (W_0^0\cdot\nabla) W_0^0 \\
&\quad + \textstyle\sum_l \beta_l^0\, b_l + \nabla(q^0 - \tfrac12 |W_0^0|^2), \\
F^{(i)} &= \tfrac{1}{S(0)}\,\partial_\theta W_0^i + \varsigma_i\,\partial_\theta W_0^0 + \tfrac12 W_0^i + \tfrac12 (y\cdot\nabla) W_0^i - \Delta W_0^i + (W_0^0\cdot\nabla) W_0^i + (W_0^i\cdot\nabla) W_0^0 \\
&\quad + \textstyle\sum_l \beta_l^i\, b_l + \nabla(q^i - W_0^0\cdot W_0^i), \\
F^{(ii)} &= \varsigma_i\,\partial_\theta W_0^i + (W_0^i\cdot\nabla) W_0^i - \tfrac12 \nabla |W_0^i|^2, \\
F^{(ii')} &= \varsigma_i\,\partial_\theta W_0^{i'} + \varsigma_{i'}\,\partial_\theta W_0^i + (W_0^i\cdot\nabla) W_0^{i'} + (W_0^{i'}\cdot\nabla) W_0^i - \nabla(W_0^i\cdot W_0^{i'})
\end{aligned}$$

for $1 \le i < i' \le n$, where β_l^i are the components of the vector β^i, and let $\langle F, G\rangle_\chi$ be the integral of $F\cdot G$ over $(0,1)\times B_{r_\chi}$. The term $F^{(ii)}$ is the diagonal quadratic coefficient. For $i < i'$, $F^{(ii')}$ already contains both orders of the cross term, so each pair occurs once in the sum in the proof below.

Lemma B.13. *Let $j \in \mathcal{J}$. Let C_j^0, C_{ji}^1, $\mathsf{C}_{jii'}^2$ and $\tilde{\mathsf{C}}_j^0$, $\tilde{\mathsf{C}}_{ji}^1$, $\tilde{\mathsf{C}}_{jii'}^2$ be numbers such that, for $1 \le i \le i' \le n$,*

$$\begin{gathered}
|\mathsf{C}_j^0 - V_j\cdot\beta^0 + \langle F^{(0)}, \psi_j\rangle_\chi| \le \tilde{\mathsf{C}}_j^0, \qquad |\mathsf{C}_{ji}^1 - V_j\cdot\beta^i + \langle F^{(i)}, \psi_j\rangle_\chi| \le \tilde{\mathsf{C}}_{ji}^1, \\
|\mathsf{C}_{jii'}^2 + \langle F^{(ii')}, \psi_j\rangle_\chi| \le \tilde{\mathsf{C}}_{jii'}^2,
\end{gathered} \tag{B.58}$$

let $\mathsf{n}_F^{\mathrm{out}} \ge \|F_\alpha^{\mathrm{a}} \mathbf{1}_{|y|>r_\chi}\|_{\mathcal{H}}$ for every $\alpha \in \mathcal{K}$ and $\mathsf{n}_j^{\mathrm{out}} \ge \|\psi_j \mathbf{1}_{|y|>r_\chi}\|_{\mathcal{H}}$, and let

$$\begin{aligned}
o_j &= |\mathsf{C}_j^0| + \tilde{\mathsf{C}}_j^0 + (|\mathsf{C}_{jj}^1 - d_j| + \tilde{\mathsf{C}}_{jj}^1)\,\hat\varrho_j + \sum_{i\ne j} (|\mathsf{C}_{ji}^1| + \tilde{\mathsf{C}}_{ji}^1)\,\hat\varrho_i \\
&\quad + \sum_{i\le i'} (|\mathsf{C}_{jii'}^2| + \tilde{\mathsf{C}}_{jii'}^2)\,\hat\varrho_i\,\hat\varrho_{i'} + \mathsf{n}_F^{\mathrm{out}}\,\mathsf{n}_j^{\mathrm{out}},
\end{aligned} \tag{B.59}$$

with $\hat\varrho_i = \varrho_i$ for $i \in \mathcal{J}$ and $\hat\varrho_i = \varrho_*/|d_i|$ otherwise. Then (7.34) holds for every $\alpha \in \mathcal{K}$.

Proof. Let $\alpha \in \mathcal{K}$. Since f_* vanishes for $|y| \le r_\chi$, the radial phase $f_\alpha(|y|^2)$ coincides in B_{r_χ} with $f(|y|^2)$, with all its derivatives. The fields W_α^i and the functions q^i depend on the phase only through the factors $E_{\iota,l}$ and the operators $\mathcal{V}_l$ of (B.5); hence in $\mathbb{R}\times B_{r_\chi}$ they coincide, with their derivatives, with W_0^i and with the functions q^i for the phase f. In the formula for F_α^{a} of Subsection B.1 we substitute $W_\alpha^{\mathrm{a}} = W_0^0 + \sum_i \alpha_i W_0^i$, $q_\alpha = q^0 + \sum_i \alpha_i q^i$, $1/S(\alpha) = 1/S(0) + \varsigma\cdot\alpha$ and $\beta^{\mathrm{a}}(\alpha) = \beta^0 + \sum_i \alpha_i \beta^i$, and we collect the powers of α: almost everywhere in $\mathbb{R}\times B_{r_\chi}$,

$$F_\alpha^{\mathrm{a}} = F^{(0)} + \sum_i \alpha_i\, F^{(i)} + \sum_{i\le i'} \alpha_i\,\alpha_{i'}\, F^{(ii')}.$$

The fields (B.57) are bounded on $(0,1)\times B_{r_\chi}$ (Subsection B.1), and

$$\begin{aligned}
V_j\cdot\beta^{\mathrm{a}}(\alpha) - \langle F_\alpha^{\mathrm{a}}, \psi_j\rangle_{\mathcal{H}} - d_j\,\alpha_j &= [V_j\cdot\beta^0 - \langle F^{(0)}, \psi_j\rangle_\chi] + \sum_i \alpha_i [V_j\cdot\beta^i - \langle F^{(i)}, \psi_j\rangle_\chi] - d_j\,\alpha_j \\
&\quad - \sum_{i\le i'} \alpha_i\,\alpha_{i'}\,\langle F^{(ii')}, \psi_j\rangle_\chi - \big\langle F_\alpha^{\mathrm{a}} \mathbf{1}_{|y|>r_\chi}, \psi_j \mathbf{1}_{|y|>r_\chi}\big\rangle_{\mathcal{H}}.
\end{aligned}$$

By (B.58) the first bracket is at most $|\mathsf{C}_j^0| + \tilde{\mathsf{C}}_j^0$ in absolute value, the coefficient of α_i is at most $|\mathsf{C}_{ji}^1| + \tilde{\mathsf{C}}_{ji}^1$ for $i \ne j$ and, with the term $-d_j\,\alpha_j$, at most $|\mathsf{C}_{jj}^1 - d_j| + \tilde{\mathsf{C}}_{jj}^1$ for $i = j$, and the coefficient of $\alpha_i\,\alpha_{i'}$ is at most $|\mathsf{C}_{jii'}^2| + \tilde{\mathsf{C}}_{jii'}^2$. Cauchy–Schwarz bounds the exterior pairing by $\mathsf{n}_F^{\mathrm{out}}\mathsf{n}_j^{\mathrm{out}}$. The parameter set (7.29) gives $|\alpha_i| \le \hat\varrho_i$: for $i \in \mathcal{J}$ this is its coordinate constraint;

for $i \notin \mathcal{J}$, retain the nonnegative summand $d_i^2\alpha_i^2 \le \varrho_*^2$ and divide by $|d_i| > 0$. The assumptions $d_i \neq 0$ and $\varrho_* \ge 0$ are those of Subsection 7.3. Substituting these bounds yields o_j in (B.59). With $\bar{\ell} \equiv 0$, the resulting inequality is (7.34). □

Impose (7.9) and (7.23) directly. For each $\alpha \in \mathcal{K}$, (7.23) bounds the $\mathcal{H}$-norm of the residual defined in Subsection B.1 and the Euclidean norm of its N pairings in (7.17). The integrals extend over $(0,1) \times \mathbb{R}^3$. Require $0 < S(\alpha) \le S_{\max}$, invertible V and the sign conditions of Subsections 7.2–7.3, including $\omega > 0$, $M > 0$ and $d_j \neq 0$. Also impose (7.21), (7.24) with $\bar{\ell} = 0$, (7.25)–(7.28) and (7.35)–(7.37). In Proposition B.14 the conditions of this appendix include the full cell conditions specified at its beginning.

Proposition B.14. *Let data of the form of Subsection B.1 be given, with the weights, bounds, matrices and numbers of Subsections B.2 to B.4, and suppose that they satisfy the conditions of this appendix. Then the set Γ, the set $\mathcal{K}$ of (7.29), the periods $S(\alpha)$, the fields W_α^{a}, W^{r}, b_i, F_α^{a}, Ξ_i, e_k and ψ_j, the maps β^{a} and $\bar{\ell} \equiv 0$, the weight $w \equiv w_*$, the matrices V and Y, and the numbers of the conditions are a certificate in the sense of Section 7.*

Proof. Subsection B.1 gives the representations required in Subsections 7.2–7.3, (7.23) gives $F_\alpha^{\mathrm{a}} \in \mathcal{H}$, and the third inequality of (B.22) gives $\|e_k\|_{\mathcal{H}} \le c_0$. Fix $\alpha \in \mathcal{K}$. Proposition B.8 yields $\mathcal{I}^- < \infty$ and (B.21). Admissibility and Lemma B.2 give (7.18) with c_1, c_2 from (B.22). Lemma B.9 gives (7.19) and (7.22), and Lemmas B.10, B.12 and B.13 give (7.32), (7.33) and (7.34), respectively. Lemmas B.9, B.10 and B.12 hold throughout $\mathcal{D}_{S(\alpha)}$, hence on $\mathcal{D}^{\Gamma}_{S(\alpha)}$. The preceding paragraph imposes the remaining conditions. □

For any data satisfying Proposition B.14, Theorem 7.1 supplies the exact profile and Corollary 7.2 gives the conclusions of Theorem 1.1 on $\mathbb{R}^3$ and $\mathbb{T}^3$.

The conditions fix few constants: the weight $w_* = \frac{39}{50}$, the numbers $\frac{3}{50}$, 1000 and $400\,(1+2^{-18})$ of (B.21) and (B.22), and the constants $\sqrt{7/2}$ and $\sqrt{9/2}$ of the bounds for the stretching term, the second of which takes the place of the first for the polynomial model of $\mathbb{V}$, whose trace need not vanish. Every choice of data satisfying (B.22) has $c_1 \ge \frac{50}{3}$, $c_0 > 81.64$ and $c_2 > 129$. Hence (7.21), (7.25) and (7.27) imply $c_L > 98.3$, $M < 0.01695$ and $\varepsilon < 0.0075$, respectively. The remaining numbers are specified by the data. For these implications, Proposition B.1 gives $c_{\mathsf{M}}^2 \ge 50/3$, and (7.21) gives $c_L \ge c_1 + c_0$ since $0 \le \vartheta < 1$. Equation (7.25) gives $M < (2c_L c_w)^{-1}$. Keeping only $4\varepsilon^2/w_*$ on the left of (7.27) gives $2\varepsilon \le \sqrt{w_*}M$, which yields the last bound.

These are thresholds for *any* admissible data. No achieved residual or margin can be read from them. For example, (7.21) needs an enclosure strictly below one for $(c_1 + c_0)c_{\mathrm{r}}$, whereas equality is allowed in the residual budget (7.24). The reported computation in Section 1.5 supplies no published instance margins for these, the parameter inclusions or the cell and tree inequalities.

Appendix C. The computation of the approximate profile

The computation described by Alpöge and Buckmaster produces the field $\bar{W}$ in Figures 1–4. Their Newton solve determines its discretized coefficients (Subsections C.2–C.3). Subsection C.4 describes how they choose further fields for the estimates of Appendix B. To compare a picture with the exact solution, the phases must agree. At normalized phase θ, (7.38) bounds the distance from $W(S(\alpha)\theta)$ to $W_\alpha^{\mathrm{a}}(\theta)$ in $\dot{H}^1$. A comparison with $\bar{W}(\bar{S}\theta)$ requires the additional distance from that family member to the pictured field. The same two distances bound

the L^2 vorticity error by the solenoidal Plancherel identity. Persistence of a pictured level surface would require pointwise derivative control and a regular level.

C.1. The approximate profile. Write $\bar{S}$ for the period of $\bar{W}(s, y)$ in similarity time. In the normalized variable $\theta = s/\bar{S}$, the field has the form (B.6). Subsection C.4 describes the other fields used in (B.9). The approximate family depends affinely on α inside B_{r_χ}. Beyond that ball its radial phase also varies. Figure 2 resolves the computed field into its angular mean and the first nonzero angular harmonic of its velocity.

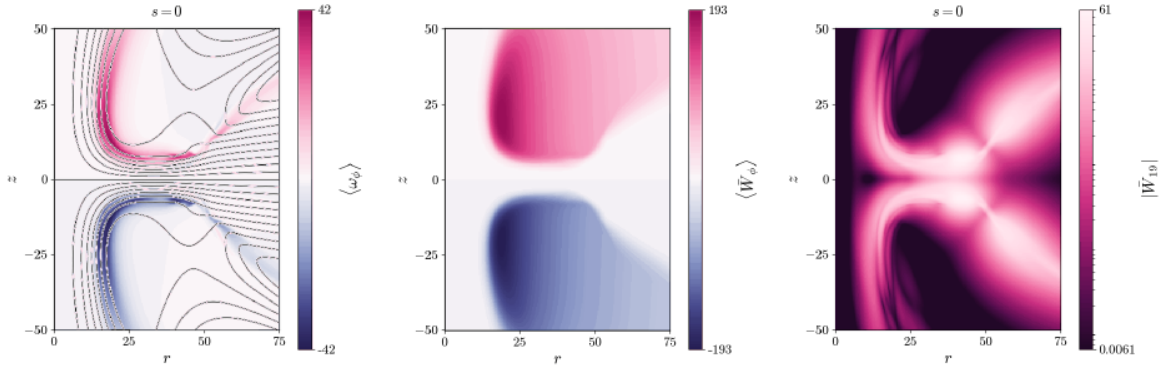


FIGURE 2. Meridian sections of the computed $\bar{W}$ at $s = 0$. Radius r runs horizontally from 0 to 75, height $z = y_3$ vertically from -50 to 50. Left, mean azimuthal vorticity (-42 to 42) with stream-function contours. Middle, mean azimuthal velocity (-193 to 193), with opposite signs across $z = 0$. Right, modulus of the $e^{19i\phi}$ velocity coefficient (0.0061 to 61, logarithmic), increasing from dark to light.

The continuation in Subsection C.3 leads to the angular wave number $k_{\mathrm{rot}} = 19$. This is the rotational order used in (B.1) for the reported example. The construction does not establish a least admissible order. The harmonics (B.2) and the coefficient relations of Subsection B.1 make $\bar{W}$ invariant under rotation by $2\pi/19$ about the y_3 axis and under R_*, the rotation by π about the y_1 axis. Advancing s by half a period rotates the field by $\pi/19$:

$$\bar{W}(s + \tfrac{1}{2}\bar{S}, y) = R_{\pi/19}\, \bar{W}(s, R_{\pi/19}^{-1}\, y) \qquad (s \in \mathbb{R},\ y \in \mathbb{R}^3).$$

For $\iota_2 = 0$, the harmonic E_ι depends on θ and ϕ through $\phi - 2\pi\theta/19$; for $\iota_1 = \iota_2$, it depends on them through $\phi + 2\pi\theta/19$. These two families give patterns turning in opposite senses, each through $2\pi/19$ per period, and R_* exchanges them. Every other harmonic is a product of one harmonic from each family. Indeed, $\iota = (\iota_1 - \iota_2, 0) + (\iota_2, \iota_2)$, and the phases in (B.2) add. The computed period is $\bar{S} \approx 0.36$.

At $s = 0$, vorticity concentrates in an axial column and two layers flanking $y_3 = 0$ (Figure 1). The non-axisymmetric velocity is comparable in size to the mean; its first harmonics $e^{\pm 19i\phi}$ are largest near the outer end of the layer (Figure 2). A horizontal section resolves the nineteen arms and their terminal vortices (Figure 3). The half-period rotation makes the angular mean $\bar{S}/2$-periodic.

For an exact profile in Theorem 7.1, testing the curl of (1.2) against $\Omega = \operatorname{curl} W$ over one period gives

$$\int_0^S \int_{\mathbb{R}^3} (\Omega \cdot \nabla) W \cdot \Omega \, dy \, ds = \int_0^S \Big(\|\nabla \Omega\|_2^2 + \tfrac{1}{4} \|\Omega\|_2^2 \Big) \, ds.$$

Periodicity removes the time derivative, and solenoidality removes transport by W. The term $\Omega + \frac{1}{2}(y \cdot \nabla)\Omega$ contributes $(1 - \frac{3}{4})\|\Omega\|_2^2$. Decay of Ω and $\nabla\Omega$ as $|y|^{-2}$ and $|y|^{-3}$ permits removal of cutoffs (Remark 1.2). Thus integrated stretching balances diffusion and the similarity term. Applying this identity to a computed field requires its residual as well.

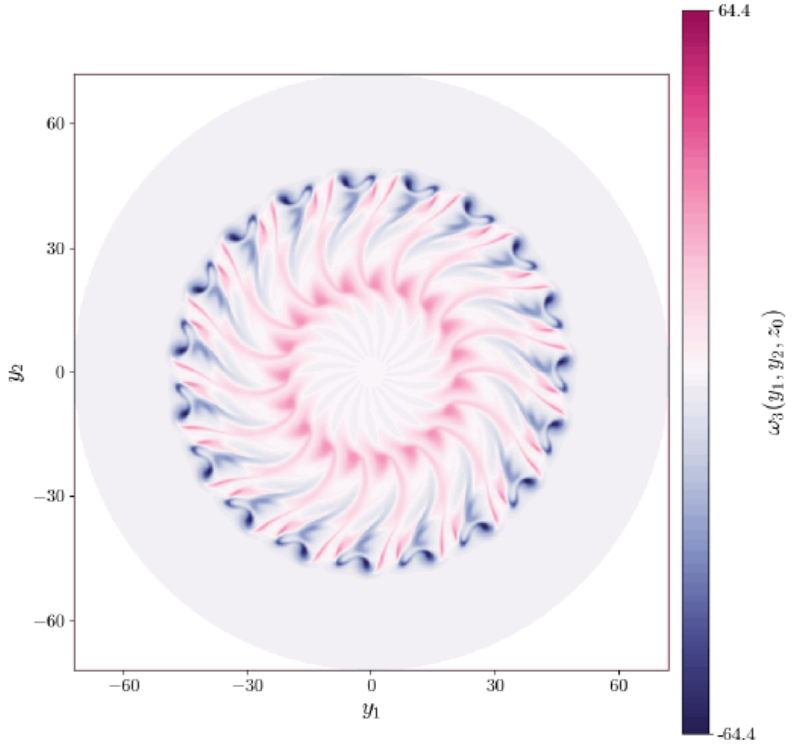


FIGURE 3. Horizontal section through the upper layer of the computed $\bar{W}$ at $s = 0$ and $y_3 = 6.5$. The horizontal coordinate is y_1 and the vertical coordinate is y_2. Magenta gives positive vertical vorticity ω_3 and blue gives negative vorticity, on a scale from -64.4 to 64.4. The nineteen arms can be counted by their tips around the rim.

The measured $\|\nabla \bar{W}(s)\|_2$ is of order 10^4 and varies by less than one part in a thousand over a period. The half-period rotation by $\pi/19$ preserves this norm even as it moves the individual spiral arms. At the origin $\bar{W} = 0$ and $\nabla \bar{W}$ is an s-dependent multiple of $\nabla(y_1, y_2, -2y_3)$ (§B.1). The three weighted suprema in Figure 4 peak between radii 10 and 100. Those for the field and its gradient suggest a constant of order 10^5 in (7.8). The decay powers themselves follow from $h = 1$ in (B.7)–(B.8) and the radial phase conditions.

In particular, $\bar{W} = O(|y|^{-1})$ at infinity. Its oscillation at large radius motivates the following choice of radial phase in (B.5)

$$f(\rho) = \frac{2\pi}{\bar{S}} \log\Big(1 + \frac{\rho}{r_c^2}\Big),$$

where $r_c > 0$. In (B.5), this function is evaluated at $\rho = |y|^2$.

Each term with harmonic E_ι has $l = \iota_1 - 2\iota_2$, the negative of its integer frequency in θ. The functions (B.5) become

$$E_{\iota,l}(\theta, y) = e^{19i\iota_1\phi} \exp\Big(\frac{2\pi i l}{\bar{S}}\Big[\log\Big(1 + \frac{|y|^2}{r_c^2}\Big) - s\Big]\Big), \qquad s = \bar{S}\theta,$$

and for large $|y|$, the bracket equals $2\log|y| - s$ plus a constant and an $O(|y|^{-2})$ term. In the Eulerian variables (1.4), $2\log|y| - s = 2\log|x| - \log T$. Hence the Eulerian rescaling (1.4) of a field of the form (B.6), with these phases and coefficients asymptotic to angular functions divided by $|y|$, converges at every fixed $x \neq 0$ as $t \to T$. For fixed $x \neq 0$, the direction of y is that of x. The phase converges, and $(T-t)^{-1/2}|y|^{-1} = |x|^{-1}$ cancels the amplitude scaling in (1.4). Representing the same far field without this phase would put its logarithmic oscillation into the radial coefficients, at frequencies proportional to l.

This limit concerns the represented field with the stated coefficient asymptotics. Subsection C.4 derives the corresponding condition on the phase when the period varies. The $\dot{H}^1$ correction bound (7.38) supplies no such pointwise expansion for the exact field.

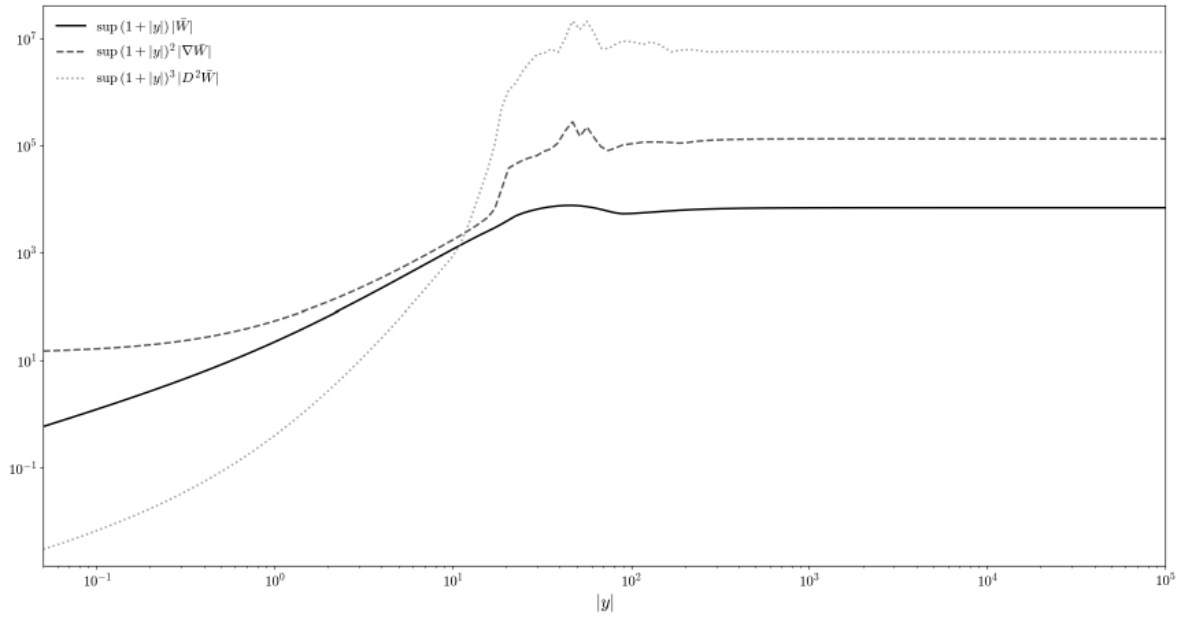


FIGURE 4. Estimated suprema over directions and one period for the computed $\bar{W}$, plotted against $|y|$ on logarithmic axes. At large radius the solid lower curve gives $(1+|y|)|\bar{W}|$, the dashed middle curve $(1+|y|)^2|\nabla\bar{W}|$, and the dotted upper curve $(1+|y|)^3|\nabla^2\bar{W}|$. A horizontal plateau indicates that the corresponding weighted quantity is approximately constant over the sampled radii.

C.2. The discretization and Newton's method. We compute with the first two equations of (1.2) written as

$$\partial_s W + \tfrac{1}{2}W + \tfrac{1}{2}(y\cdot\nabla)W + \operatorname{curl} W \times W + \operatorname{curl}\operatorname{curl} W + \nabla H = 0, \qquad \operatorname{div} W = 0, \tag{C.1}$$

where $H = q + \frac{1}{2}|W|^2$. For smooth solenoidal W, the identities $\operatorname{curl}\operatorname{curl} W = -\Delta W$ and $\operatorname{curl} W \times W = (W \cdot \nabla)W - \nabla(|W|^2/2)$ give this form. The field has the form (B.6), with $k_{\mathrm{rot}} = 19$, the radial phase above, and $h_0 = 2$, $h = 1$. The unknowns are the piecewise polynomial coefficients of A, B, S and T in (B.7)–(B.8), the numbers γ for finitely many harmonics, and, at the stages specified in Subsection C.3, the period. All harmonics use one partition in (σ, η): radial rings, the last extending to infinity, and intervals in $\eta \geq 0$. The R_* relations of Subsection B.1 determine $y_3 < 0$ and impose conditions at $\eta = 0$. On each rectangle, a harmonic has its own degrees in σ and η, or is absent. The basis functions satisfy the joining orders in Subsection B.1. This representation is divergence-free, has the axis, origin and far-field behaviour prescribed by the factors in (B.7)–(B.8), and satisfies the symmetries.

We impose (C.1) weakly, integrating over one period and $\mathbb{R}^3$ against fields on the same partition, of the same form with $h = 3$ in place of $h = 1$. There are as many test fields as unknown coefficients. The test fields are divergence-free, so the pressure term drops out. We evaluate products in $\operatorname{curl} W \times W$ on a grid in the two harmonic angles and use Gauss quadrature on each rectangle. When the period is unknown, its derivative adds a column to the Newton system and a phase condition fixing translation in s adds a row. To judge the current iterate, we evaluate the left-hand side of (C.1) without ∇H on a larger set of harmonics, subtract a gradient fitted by least squares, and measure the remainder in $\mathcal{H}$ of Subsection 7.1. For this iterate the measured norm approximates the minimum obtained by varying the pressure coefficients. For the represented family, (7.23) requires bounds on the specified F_α^{a} over the whole parameter set, including its pairings in (7.17).

Newton's method solves the discrete equations. Differentiation with respect to the coefficients gives the weak form of

$$v \mapsto \partial_s v + \tfrac{1}{2} v + \tfrac{1}{2}(y \cdot \nabla)v + \operatorname{curl} W \times v + \operatorname{curl} v \times W + \operatorname{curl}\operatorname{curl} v \tag{C.2}$$

at the current field W. Flexible GMRES uses only matrix-vector products. The preconditioner solves blocks of neighbouring harmonics, sometimes restricted to low polynomial degrees. The block factorizations use nested dissection over the rectangles. The authors report at most five Newton steps per solve. The starting point was continued from a nearby solution or transferred from the preceding discretization. The number of Newton steps measures the work on a single discretization. The error enclosures concern the continuum equation and the family in (B.9).

C.3. Continuation and refinement. The starting field for Newton's method is obtained by continuation, first with finite differences and then with spectral elements. These discretizations precede that of Subsection C.2. The continuation uses the family

$$\partial_s U + (1 - g)\, U + g\, (y \cdot \nabla)U + (U \cdot \nabla)U + \nabla \pi = \nu\, \Delta U + F, \qquad \operatorname{div} U = 0,$$

with scaling parameter g, viscosity $\nu > 0$ and force F. The target is $g = \frac{1}{2}$, $\nu = 1$, $F = 0$.

One rotating pattern. We begin with an axisymmetric steady state at g near 0.502 and with a small force. It is computed from a field with swirl changing sign across $y_3 = 0$, an inward-flowing layer along that plane, a column, and an axial jet. On the layer we superpose finite-amplitude waves of azimuthal wave numbers 14 to 20, turning rigidly about the axis. We reduce the force to zero along with the damping introduced at this step. We then work in $y_3 > 0$ with one wave family and allow its azimuthal wave number to be real. Among the unforced families found by continuation, the first stays in $g > \frac{1}{2}$ and a second reaches $g = \frac{1}{2}$. At that value we have an unforced state with the desired scaling.

Two counter-rotating patterns. We superpose this state with its oppositely turning image under R_* and impose the exact condition at $y_3 = 0$. Further refinements use viscosity in place

of g as an unknown and bring the wave number to 19. We finally project the result onto a space of the form used in Subsection C.2.

Fixing viscosity and period. Starting from this field, we use nine successive discretizations of Subsection C.2, transferring each computed field to the next set of degrees. In the first three, viscosity and period are unknown, and we normalize the mean swirl. For the continuum equations, solutions at different viscosities are dilations of one another; the normalization chooses one. For $g = \frac{1}{2}$ and $F = 0$, the dilation is $U_\nu(s, y) = \sqrt{\nu}\, U_1(s, y/\sqrt{\nu})$, with $\pi_\nu(s, y) = \nu\pi_1(s, y/\sqrt{\nu})$ in the continuation equation. We bring the viscosity to one along this family. The fourth discretization fixes it at one and solves for the period. That period is $\bar{S}$ and is fixed in the last five discretizations. The ninth field is $\bar{W}$.

Between discretizations we choose degrees separately for each harmonic and rectangle. One rule raises or lowers them using residual size and coefficient decay, adding omitted harmonics wherever they contribute to the residual. The other uses coefficient decay weighted by the estimated local rates of the equation's terms. Far out, the radial phase removes the spiral oscillation. The remaining frequencies in $\log|y|$ set the radial degrees, and all rings use the same degrees in η.

C.4. Further fields of the certificate. We compute b_i and the constraint fields Ξ_i in (7.11) from the equation linearized about one field in the sequence of Subsection C.3. Perturbations are discretized more coarsely than that field. For a given right-hand side, we solve with (C.2), bordered by the period derivative and a condition removing translation in s. We measure perturbations in a norm of the form (7.14) with constant weight $w > 0$, and right-hand sides in $\mathcal{H}$. The block Lanczos iteration searches for directions with the largest ratio of solution norm to right-hand-side norm. It returns candidate right-hand sides together with their solutions. We retain finitely many of these pairs. There is one further pair for time translation, whose perturbation is the s-derivative of the field. In each pair, the right-hand side gives b_i and the solution v_i gives the constraint through $w\, v_i - \Delta v_i$. For a divergence-free field v,

$$\big\langle \nabla v, \nabla(w\, v_i - \Delta v_i)\big\rangle_{\mathcal{H}} = \langle \Delta v, \Delta v_i\rangle_{\mathcal{H}} + w\, \langle \nabla v, \nabla v_i\rangle_{\mathcal{H}}$$

is the inner product of v and v_i in this norm. For the right-hand side defining b_i and the constraint vector $w\, v_i - \Delta v_i$, we choose the radial cutoff and smoothing so that both fields remain divergence-free and retain their symmetries. For Φ_i and Ξ_i of Subsection B.1 we repeatedly quarter a meridional cell. Up to normalization, Φ_i is r times a least-squares spline with coefficients in integer multiples of a power of 2. The rounded cellwise polynomial Ξ_i approximates $\nabla\Phi_i$. The bound c_Ξ controls $\|\sum_i x_i(\Xi_i - \nabla\Phi_i)\|_{\mathcal{H}}$ for every unit vector $x \in \mathbb{R}^n$. Its contribution $\mathsf{n}_Y c_\Xi/\sqrt{w_*}$ to (B.53) enters a_j in the parameter bound (7.35).

Lanczos locates directions amplified by the discretized inverse and suggests forcing fields and constraints. The uniform bound for the full linearization still requires (7.18), including its treatment of unrepresented harmonics and the exterior.

The fields W^i in (B.9), $1 \le i \le n$, together with β^i and ς_i, approximately solve the linearized equation with prescribed constraint values. The unknowns include the coefficients of a force and a change of period. The equation for the first-order residual is $F^{(i)} \simeq 0$, with $F^{(i)}$ as in (B.57). Here $\simeq$ means equality modulo gradients as defined in (7.2); the numerical approximation concerns satisfaction of that equation. The domain in Subsection 7.1 controls the sum of time differentiation and spatial drift. For an oscillatory tail, the period and radial phase enter this sum together. For $\rho = |y|$ and a nonzero integer l, the leading scalar mode $g_l = \rho^{-1}\exp(il[c\log\rho - 2\pi\theta])$ obeys

$$(S^{-1}\partial_\theta + \tfrac{1}{2}\rho\partial_\rho + \tfrac{1}{2})g_l = il(c/2 - 2\pi/S)g_l.$$

For this nonzero oscillatory tail the coefficient must vanish to remove the ρ^{-1} term, since $\int_1^\infty \rho^{-2}\rho^2\,d\rho = \infty$. Suppose the base slope is $c(0) = 4\pi/S(0)$. In (B.9) the slope of $\log\rho$ becomes $c(0)+\mathsf{k}\cdot\alpha$, while $4\pi/S(\alpha)$ changes by $4\pi\varsigma\cdot\alpha$. Cancellation along the family therefore requires $(\mathsf{k}-4\pi\varsigma)\cdot\alpha = 0$ at each parameter carrying this leading tail. This calculation explains the choice of phase in the computation. Inside the ball $f_* = 0$, so (B.9) depends affinely on the parameter. Lemma B.13 consequently expands the interior residual to second order and uses a separate norm bound outside. Together these estimates must give (7.23) throughout the parameter set.

The adjoint fields ψ_j approximately solve the adjoint linearized equation. As in (7.33), the right-hand sides are combinations of the fields representing the constraints. For each adjoint field ψ_j, the vector of its pairings with the forcing fields b_i is prescribed and compared with the column V_j in (7.35). Their radial decay exponent in (B.7)–(B.8) is $h = 3$.

We use the rates $\nu_p = \pm 2\pi/(19\bar{S})$ of the computed patterns to choose the multiplier weights λ_p, μ_p. At period $\bar{S}$, (B.15) gives $4\pi\iota_2/\bar{S}$ in the first frame and $-4\pi(\iota_1-\iota_2)/\bar{S}$ in the second. At period $S(\alpha)$ one must add $2\pi(2\iota_2-\iota_1)(S(\alpha)^{-1}-\bar{S}^{-1})$ to each expression. Admissibility in Subsection B.2 requires the frequency bounds for every $\alpha \in \mathcal{K}$. The multiplier supplies positive terms proportional to $\lambda_p|\Omega_p(\iota)|$ and $\mu_p|\Omega_p(\iota)|$ in (B.16). This uses the nonnegative weights and the positive-frequency condition on each $\mathcal{N}_p$ from Subsection B.2; the sign in (B.14) treats $-\mathcal{N}_p$. These terms must absorb errors including transport by $W^{\mathrm{r}} + \frac{1}{2}y - \nu_p\,\mathbf{e}_z \times y$, the velocity relative to that frame. Considering only this error and omitting weight derivatives, the best pointwise weight is proportional to frequency divided by the square of this speed. For positive fixed auxiliary factors and nonzero speed, this follows by maximizing the linear frequency gain minus the quadratic transport error as a function of the weight. The weights are splines in (B.3). Numerical optimization chooses their coefficients to maximize the least, over cells and harmonic classes, of the suitably normalized differences between the two sides of the inequalities containing them.

Acknowledgments

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Original authors' statement on AI use. We used Anthropic models (Claude) throughout this work, and no others. The work builds on unreleased work of the authors on computer-assisted proofs, in particular concerning the non-uniqueness of Leray–Hopf solutions in the style of Jia and Šverák [65] and blow-up for fluid equations with boundary. It builds also on numerical work of the authors on finding self-similar profiles, on our earlier papers on singularity formation, and on our unreleased drafts on the forced hypodissipative Navier–Stokes equations and the forced Navier–Stokes equations. The idea of pursuing discretely self-similar solutions came from the paper of the second author with Coiculescu and Li on non-uniqueness [17]. The agents were given a curated dossier of potential leads and then guidance after our strictly-uncontaminated experiment producing unforced Euler and forced Navier–Stokes blow-up from our forced Euler and forced hypodissipative Navier–Stokes manuscripts. A simpler mechanism sufficed to produce unforced hyperdissipative Navier–Stokes blow-up, i.e. for $\nu\cdot(-\Delta)^{\frac{501}{500}}$, and then a heavier computation produced this profile. We used the enslopification skills of Scott Armstrong [10] to create a draft from an initial Lean proof (Subsection 1.5); the draft followed

that proof step by step. We then rewrote it with Claude under our direction, citing standard results in place of proofs and simplifying the argument over successive revisions, so that the proof given here no longer follows the Lean proof.

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