FINITE-TIME SINGULARITY FORMATION FOR THE THREE-DIMENSIONAL EULER EQUATIONS

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Abstract. We construct smooth odd initial data on the three-dimensional torus whose solution of the unforced incompressible Euler equations develops a finite-time singularity. The vorticity magnitude at the stationary particle at the origin tends to infinity, while the velocity converges on the whole torus in every spatial C^α norm with $0 < \alpha < 5/6$. Away from the origin the velocity and its first time derivative converge locally in every spatial C^m norm. Each new wave has small velocity but a large gradient across its phase planes. The preceding flow amplifies the wave and turns its normal. After the amplitude peaks, the wave number is large enough to leave a strong shear in the orientation needed by the next wave. The launch times accumulate in finite time and the velocity increments are summable. To cancel the phase-averaged error of localization, we choose an initial velocity correction that fixes each particle's final position to first order. A steep central slope in the wave profile supplies the shear. A shallow return slope bounds the positive pressure curvature in this endpoint problem.

Reading edition v8, 22 November 2026. Will Cook adapted the authors' 20 November manuscript (revision fd3a158) with AI assistance. The proof and numbered results are the authors'. Section 8.4 of this edition derives further terminal estimates. The authors' Lean correspondence identifies the formal statements for the numbered results of their manuscript.

1. Introduction

On the three-dimensional torus $\mathbb{T}^3$, represented by the cube $[-1/2, 1/2]^3$ with opposite faces identified, the velocity u and pressure p of an incompressible fluid satisfy the unforced Euler equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \qquad \operatorname{div} u = 0 \quad \text{on } \mathbb{T}^3 \times [0, T_*). \tag{1.1}$$

For smooth divergence-free initial data the local theory gives a unique smooth solution on a maximal interval $[0, T_*)$ [13, §15.2(i),(iii), pp. 150–151]. We construct odd initial data for which $T_* < \infty$. Oddness fixes the particle at the origin, where its velocity is always zero. The vorticity $\omega = \operatorname{curl} u$ at that particle grows without bound even as the velocity converges uniformly on the entire torus.

Theorem 1.1. *There is an odd divergence-free $u_0 \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ whose maximal classical solution u of (1.1) is smooth and odd on $\mathbb{T}^3 \times [0, T_*)$ for a finite $T_* > 0$, and satisfies*

$$\int_0^{T_*} \|\nabla u(t)\|_{L^\infty} \, \mathrm{d}t = \infty, \qquad \int_0^{T_*} \|\operatorname{curl} u(t)\|_{L^\infty} \, \mathrm{d}t = \infty. \tag{1.2}$$

Moreover,

$$\sup_{0 \le t < T_*} \|u(t)\|_{L^\infty} < \infty, \qquad \lim_{t \uparrow T_*} |\operatorname{curl} u(t, 0)| = \infty. \tag{1.3}$$

For every compact $K \subset \mathbb{T}^3 \setminus \{0\}$ and every integer $m \ge 0$, both $u(t)$ and $\partial_t u(t)$ converge in $C^m(K)$ as $t \uparrow T_*$.

One wave and its successor. At the origin oddness gives $u(t, 0) = 0$. Taking the curl of (1.1) there gives $\partial_t \omega(t, 0) = (\nabla u(t, 0))\omega(t, 0)$. A wave can change this gradient substantially while adding little velocity. If its velocity is $\mathbf{a}F(k \cdot x)$ with $F'(0) = 1$ and $\mathbf{a} \perp k$, its gradient at the origin is $\mathbf{a} \otimes k$ and its curl is $k \times \mathbf{a}$. Shortening the wavelength increases the gradient at a fixed amplitude.

The rank-one gradient annihilates its own curl. Its role in the construction is to act on a *different* wave.

Start with a stationary periodic shear. Add a wave whose phase planes pass through a small neighborhood of the origin. The preceding velocity gradient amplifies the wave and turns the normal to its planes. The wave vector changes both direction and magnitude. In the strong-shear reference solution, its length grows fast enough that its product with the amplitude keeps increasing even after the amplitude peaks. For the actual wave, a comparison estimate places a crossing of the prescribed coupling ratio after the reference peak and bounds the shear there. This is where the successor starts.

The wave equations along one particle determine neither a divergence-free velocity near it nor the pressure throughout the torus. We localize the wave, then correct the resulting error. For its average over one period of the phase we select the initial velocity correction by fixing the particles' end positions to first order. This choice changes the initial data even for waves that start later. The remaining errors can be made small enough to repeat the step. The launch intervals have finite total length, and the velocity increments are summable. On each interval between launches, projection onto the newest normal eliminates the newest leading curl and detects the preceding one. This gives vorticity divergence throughout the approach to the singular time.

The velocity at the singular time. The convergence in $C^\alpha(\mathbb{T}^3)$ for $0 < \alpha < 5/6$ is derived in Section 8.4 from the sizes of the velocity increments and their first derivatives. Jaswal obtains the same convergence [57, Theorem 1.5(a)]. The bounds between launches also give $\int_0^{T_*} |\omega(t,0)|\,\mathrm{d}t = \infty$ at the fixed particle. For comparison, the global integrals in (1.2) already follow from finite maximal lifespan by the periodic Beale–Kato–Majda criterion [9, p. 62, Theorem and Corollary][11, Theorem 1].

Remark 1.2. *The solution in Theorem 1.1 may be chosen so that it extends as a smooth field in space and time through T_* away from the origin. A proof is given in the auxiliary note* [31, Theorem 1.1]*, where the Euler equations are imposed only for $0 \le t < T_*$.*

1.1. **How a wave amplifies its successor.** Consider first a flow on $\mathbb{R}^3$ of the form $U(t,x) = D(t)x$. It solves the Euler equations if the smooth matrix D has trace zero and $D' + D^2$ is symmetric. Indeed, the acceleration is $(D' + D^2)x$, the negative gradient of $p_D = -x \cdot (D' + D^2)x/2$. Craik and Criminale [12, §2, (2.4)–(2.10)] superposed an exact plane wave on such flows. In the inviscid case its velocity takes the form

$$w(t,x) = \mathbf{a}(t)F(k(t) \cdot x), \qquad \mathbf{a} \cdot k = 0.$$

Here F is smooth, odd and 1-periodic, with $F'(0) = 1$, for example $F(s) = \sin(2\pi s)/(2\pi)$. At any one time the planes $k \cdot x = s$ carry a common velocity $\mathbf{a}F(s)$, tangent to each plane. A displacement of length $1/|k|$ in the direction $k/|k|$ advances the phase by one period. Thus the speed is at most $|\mathbf{a}|\|F\|_\infty$, whereas the velocity gradient at the origin is $\mathbf{a} \otimes k$.

The wave advects no part of its own velocity, since motion along $\mathbf{a}$ leaves the phase unchanged. Transporting the phase by Dx and balancing the remaining acceleration against the pressure gives

$$k' = -D^T k, \qquad \mathbf{a}' = -D\mathbf{a} + p_{\mathrm{w}} k, \qquad p_{\mathrm{w}} = \frac{2k \cdot D\mathbf{a}}{|k|^2}. \tag{1.4}$$

Differentiating $k \cdot \mathbf{a} = 0$ gives the pressure coefficient. With $P' = F$ the wave pressure is $-p_{\mathrm{w}} P(k \cdot x)$, and $Dx + w$ is an exact Euler velocity. The pressure force is parallel to k, so it does no instantaneous work on the amplitude. Thus $\frac{1}{2}(d/dt)|\mathbf{a}|^2 = -\mathbf{a} \cdot D\mathbf{a}$. The pressure force turns the amplitude, maintaining its orthogonality to the wave vector. For related instabilities of planar flows see Pierrehumbert [35] and Bayly [36].

A prescribed gradient for two waves. We now use (1.4) as an ordinary differential equation for a prescribed matrix. No Euler equation for the linear velocity associated with that matrix

is assumed. Write $k_p, \mathbf{a}_p$ for the normal and amplitude of the preceding wave. At the origin its gradient is $\sigma E_2 \otimes E_1$, where

$$E_1 = k_p/|k_p|, \quad E_2 = \mathbf{a}_p/|\mathbf{a}_p|, \quad E_3 = E_1 \times E_2, \qquad \sigma = |k_p|\,|\mathbf{a}_p|.$$

Thus E_1 is normal to the phase planes, E_2 is the velocity direction within them, and $k_p \times \mathbf{a}_p = \sigma E_3$ is the preceding wave's vorticity at the origin. The older gradient B drives $(k_p, \mathbf{a}_p)$ and hence this moving frame. The current wave feels $D = B + \sigma E_2 \otimes E_1$. To isolate the two couplings that drive it we hold the following entries of B constant in the moving frame,

$$E_1 \cdot BE_2 = \nu > 0, \qquad E_3 \cdot BE_2 = \eta > 0, \qquad \varepsilon = \eta/\nu \ll 1$$

and set the other entries to zero. Section 2 controls the effect of departures from these coefficients at the central particle of the periodic flow.

Amplification. At launch $k(0) = \lambda E_3$. Its amplitude lies in the (E_1, E_2)-plane. In the rotating frame the two components $a_i = \mathbf{a}\cdot E_i$ initially satisfy $\dot a_1 = -2\nu a_2$ and $\dot a_2 = -(\sigma-\nu)a_1$ (Section 2.2.1). The rotation of E_1 contributes the second $-\nu a_2$ in $\dot a_1$. For $\sigma > \nu$ the growing eigenvalue of this frozen system is $\sqrt{2\nu(\sigma-\nu)}$. With strong shear $S = \sqrt{\sigma/(2\nu)} \gg 1$, the growing eigendirection is $E_2 - S^{-1}E_1$ to leading order. We launch the new amplitude in that direction.

Turning. Suppose for a moment that $\eta = 0$. Then E_3 and $k = \lambda E_3$ stay fixed and the two-component system above admits exponential growth. Yet $D\mathbf{a}$ stays in the (E_1, E_2)-plane, so $k\cdot D\mathbf{a} = 0$. The growing wave supplies no first coupling for a successor. With $\eta > 0$ the normal develops a component towards $-E_2$ in the moving frame, which the shear turns towards E_1. Measure elapsed time in units of the growth time by putting $\tau = \sqrt{2\sigma\nu}\,t$. As $S \to \infty$ with ε and the normalized time interval fixed, the components of k/λ in the preceding frame tend to $(\tan\varphi, 0, 1)$, where

$$\tan\varphi = \tfrac{1}{2}\varepsilon\tau^2, \qquad 0 \le \varphi < \tfrac{\pi}{2}.$$

Thus φ is the angle from E_3 towards E_1 of the limiting normal. At finite shear the actual projected normal has slope $(k\cdot E_1)/(k\cdot E_3)$, which the comparison estimate in Section 2 controls.

Turning changes the pressure force in the amplitude equation. The coefficient $\cos^2\varphi(1-\varepsilon\tan\varphi)$ that drives the initial growth changes sign at $\tau = \sqrt{2}/\varepsilon$. The E_2 component of the reference amplitude, denoted a_{ref} and equal to one at launch, continues to grow. It reaches its maximum when its logarithmic derivative vanishes, at a time τ_m with $2 \le \varepsilon\tau_m \le 3$ (Lemma 2.5). Figure 1 shows this delay. To transfer the gain to a wave at finite shear, Section 2 measures its amplitude relative to a_{ref}. Under the comparison hypotheses on an interval containing τ_m, its maximum is at least a fixed multiple of $\exp(\varepsilon^{-1/2})$ times its launch value (Lemma 2.6 and Proposition 2.8).

Transfer to the successor. The present wave will supply the shear and its parent matrix D will play the role now held by B. We therefore measure D in the present wave's frame.

$$E_{1,+} = k/|k|, \quad E_{2,+} = \mathbf{a}/|\mathbf{a}|, \quad E_{3,+} = E_{1,+} \times E_{2,+}, \qquad \nu_+ = E_{1,+}\cdot DE_{2,+}, \quad \eta_+ = E_{3,+}\cdot DE_{2,+}.$$

At launch the small $-S^{-1}E_1$ component in the amplitude reverses the sign of η_+. With zero tilt one gets $-\nu$. With the chosen tilt one gets $(-\nu + \sigma S^{-2})/(1+S^{-2}) > 0$, since $\sigma = 2\nu S^2$. A vanishing tilt can thus determine whether another amplification is possible.

Prescribe a smaller ratio $0 < \varepsilon_+ \le \varepsilon/32$ for the successor. We can launch it as soon as $\eta_+/\nu_+ = \varepsilon_+$ in the interval specified below. First, ν_+ stays positive throughout amplification, which gives the pressure coefficient $p_{\mathrm{w}} = 2\nu_+|\mathbf{a}|/|k|$ its sign. At late times ν_+ is close to ν and η_+/ν_+ is comparable to $\cos\varphi$ (Lemma 2.7). Under the hypotheses of Proposition 2.8 the ratio is still above its target at the reference peak and below it at $2(\varepsilon\varepsilon_+)^{-1/2}$. Continuity gives a crossing after the reference peak. There the increased normal length offsets the decrease relative to the maximal amplitude $A = \max_{[0,T]}|\mathbf{a}|$ on the comparison interval.

$$\frac{|\mathbf{a}|}{A} \asymp \sqrt{\frac{\varepsilon_+}{\varepsilon}}, \qquad |k| \asymp \frac{\lambda}{\varepsilon_+}, \qquad |\mathbf{a}|\,|k| \asymp \frac{\lambda A}{\sqrt{\varepsilon\varepsilon_+}}.$$

Here $\asymp$ allows absolute constant factors. The product in the last estimate is the shear available to the successor.

For the reference solution one can see directly why the amplitude peak is too early to judge the shear. Its normal has length $\lambda \sec\varphi$. From (2.14),

$$a_{\mathrm{ref}} \sec\varphi = \frac{1}{\sqrt{\cos\varphi}} \exp\left(\int_0^\tau (1+e)\cos\varphi \, \mathrm{d}\tau' \right).$$

Here $e \geq 0$ and φ increases, so the reference shear continues to increase after a_{ref} peaks. For the finite-shear wave we use the bounds at a crossing in Proposition 2.8.

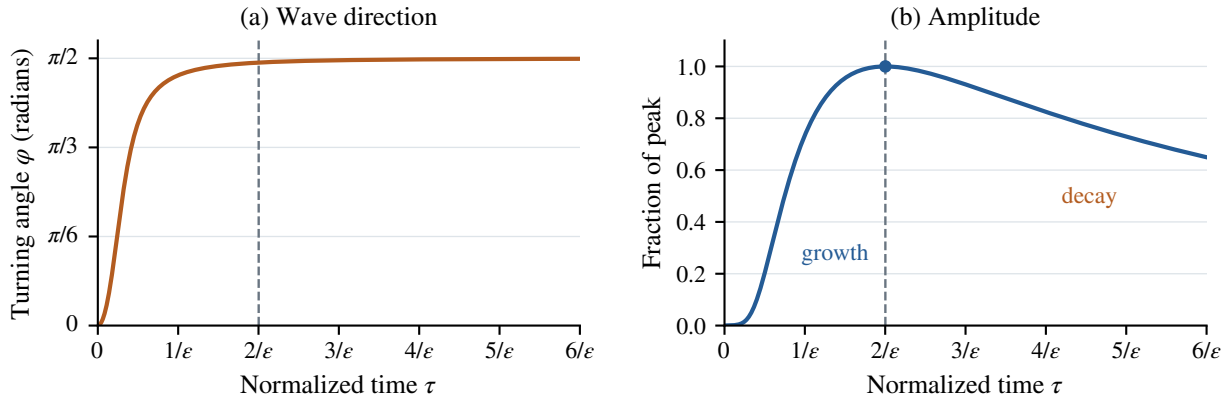


FIGURE 1. Turning and growth in the strong-shear reference system, $\varepsilon = 0.05$. The left panel gives the angle from E_3 towards E_1. The right gives $a_{\mathrm{ref}}(\tau)/a_{\mathrm{ref}}(\tau_m)$. The driving coefficient changes sign at $\varepsilon\tau = \sqrt{2}$. The amplitude reaches its peak later, at the dashed line $\varepsilon\tau_m \simeq 2.002$. Both panels show the reference solution of (2.5).

To transfer the affine calculation to the torus, localize the wave near the origin. Give each particle its phase and its cutoff value at time zero, then transport them by the flow of the preceding periodic velocity. The cutoff vanishes near the faces of a fundamental cell in the particle labels. Hence the wave extends smoothly across opposite faces even when its initial normal has nonintegral components. Transport of phase along a given flow also occurs in the short-wave stability equations of Lifschitz and Hameiri [37, §II, (7), (9), (17)] and of Friedlander and Vishik [38, (5)–(7)]. The broader asymptotic-wave theory goes back to Lax [39] and Choquet-Bruhat [40, §§2–3, pp. 3–6]. Le Dizès and Leblanc [58, §2] identified a difficulty with iterating plane waves over a general base flow. The equations for a wave along one trajectory do not give a global solution when its coefficients depend on the particle label unless the extra spatial derivative terms also cancel. Here we differentiate the amplitude, transported phase and cutoff across labels and correct the resulting divergence and momentum errors.

For the starting shear $U_0(x) = \sin(2\pi x_2)(e_1 + \rho e_3)$, let e_i be the coordinate vectors and take $\rho > 0$ small. In the initial frame of the first constructed wave, whose normal points along e_1 and whose amplitude points along e_2, the gradient $\nabla U_0(0)$ has $\nu = 2\pi$ and $\eta = 2\pi\rho$. Section 6 uses this first wave to start the successive amplifications.

1.2. **Controlling corrections in an amplifying flow.** The parent flow can amplify the response to a force even when the velocity correction starts from zero. For the phase average of the error, we choose the initial velocity correction so that the particles' end positions agree to first order with those of the parent. The resulting inverse must grow at most polynomially in the deformation and coefficient norms of the parent.

For a parent at rest and a divergence-free force $f(t,x)$, the velocity correction $b(t,x)$ satisfies $b_t = f$, and the endpoint condition is $\int_0^T b(t)\,\mathrm{d}t = 0$ at each $x \in \mathbb{T}^3$. Integration gives

$$b(0) = -\frac{1}{T}\int_0^T (T-t) f(t)\,\mathrm{d}t.$$

Even a force supported near T enters the weighted integral for $b(0)$. For a moving parent the displacement must be computed along its trajectories.

Let $X(t,a)$ be the flow of a smooth Euler solution u,p on $\mathbb{T}^3 \times [0,T]$, so $X_t = u(t,X)$ and $X(0,a) = a$. For a smooth divergence-free correction b, let X_δ be the flow of $u + \delta b$ from the same labels. Write $X_\delta = X + \delta\xi + O(\delta^2)$. Differentiating its flow equation at $\delta = 0$ gives

$$\partial_t \xi(t,a) = \nabla u(t, X(t,a))\xi(t,a) + b(t, X(t,a)), \qquad \xi(0,a) = 0. \tag{1.5}$$

All particles start at the same positions, but $\xi_t(0,a) = b(0,a)$ is free. Imposing $\xi(T,a) = 0$ selects this initial velocity.

With force f the velocity correction b and pressure variation q satisfy

$$\partial_t b + (u\cdot\nabla)b + (b\cdot\nabla)u + \nabla q = f, \qquad \operatorname{div} b = 0. \tag{1.6}$$

For the first mean correction f cancels the parent residual and the phase average of the wave error. Differentiating (1.5) along a particle and substituting (1.6) gives the displacement equation

$$\begin{gathered} \xi_{tt} + H\xi + (\nabla q)\circ X = f\circ X, \qquad H = (D_x^2 p)\circ X, \\ \operatorname{div}_x(\xi \circ X^{-1}) = 0, \qquad \xi(0) = \xi(T) = 0. \end{gathered} \tag{1.7}$$

The bound $H \le \mathrm{Id}$ means $z\cdot H(t,a)z \le |z|^2$ at every label and time. The divergence condition expresses volume preservation for the displacement in physical coordinates. For $f = 0$ this is the Jacobi equation in Arnold's geodesic description of Euler flow [5, Section 10].

For a solution of (1.7), testing with $-\xi$ and integrating by parts gives the identity

$$\underbrace{\int_0^T \int_{\mathbb{T}^3} \left(|\xi_t|^2 - \xi\cdot H\xi\right)\,\mathrm{d}a\,\mathrm{d}t}_{\mathcal{Q}(\xi)} = -\int_0^T \int_{\mathbb{T}^3} (f\circ X)\cdot\xi\,\mathrm{d}a\,\mathrm{d}t. \tag{1.8}$$

Indeed, changing variables $x = X(t,a)$ changes the pressure pairing to $\int_{\mathbb{T}^3} \nabla q\cdot(\xi\circ X^{-1})\,\mathrm{d}x = 0$. On volume-preserving paths with both endpoint maps fixed, $\mathcal{Q}$ is the second variation of the kinetic action. The family X_δ above identifies its tangent displacement ξ and fixes the terminal map to first order.

If $H \le \mathrm{Id}$ and $T \le 1$, then $|\xi(t,a)|^2 \le (T-t)\int_t^T |\xi_\tau(\tau,a)|^2\,\mathrm{d}\tau$ gives

$$\mathcal{Q}(\xi) \ge \left(1 - \frac{T^2}{2}\right)\|\xi_t\|^2_{L^2_{t,a}} \ge \frac{1}{2}\|\xi_t\|^2_{L^2_{t,a}}. \tag{1.9}$$

A negative eigenvalue of H improves the energy in its eigendirection. For example, take $y'' - \mu^2 y = f$ with $\mu > 0$. The solution $e^{\mu t}$ grows in the homogeneous initial-value problem, while testing on paths with $y(0) = y(T) = 0$ gives the positive form $\int_0^T (|y'|^2 + \mu^2|y|^2)\,\mathrm{d}t$. For Euler the pressure constraint couples the particles, but the upper bound on H still suffices for (1.9). With $f = 0$ the energy identity forces $\xi_t = 0$, hence $\xi = 0$ and $b = 0$ by (1.5). Thus the endpoint map $b(0) \mapsto \xi(T)$ is injective. Existence with the norms used in the iteration is the content of Theorem 4.2.

Brenier used a Dirichlet Poincaré inequality for paths with matching endpoints and an upper pressure-Hessian bound to obtain a criterion for action minimization [32, Theorem 5.1 and Lemma 5.2, pp. 242–244]. See also [33, Theorem 2.4, author version].

A finite step gives an approximate velocity U with pressure p_U and residual $R_U = \partial_t U + (U \cdot \nabla)U + \nabla p_U$. Along its flow X the displacement coefficient becomes $H^{\#} = (D_x^2 p_U - \nabla R_U) \circ X$ (Section 4). Its quadratic form sees only its symmetric part. For $0 < T \leq 1$ and $H^{\#} \leq \mathrm{Id}$ in this sense, Theorem 4.2 constructs a smooth correction for every smooth force. Appendix A bounds the inverse in spatial Hölder norms by a fixed power of the parent's deformation and coefficient norms. A large negative pressure curvature is allowed, but at each step we must control the positive curvature added by the wave and the gradient of the residual left behind.

In the affine wave of Subsection 1.1, the Hessian of $-p_{\mathrm{w}} P(k \cdot x)$ is $-p_{\mathrm{w}} F'(k \cdot x) k \otimes k$. The coupling ratio ε controls how the wave turns and grows. We now choose a different parameter h, the width in phase of the steep part of an odd periodic profile F_h. We require

$$F_h'(0) = 1, \qquad -Ch \leq F_h'(s) \leq 1. \tag{1.10}$$

The unit slope retains the full shear $\mathbf{a} \otimes k$ at the origin. In the normal direction $k/|k|$ the wave pressure has curvature $-p_{\mathrm{w}}|k|^2 F_h'(k \cdot x)$, and it has zero curvature in the tangent directions. When $p_{\mathrm{w}} > 0$ its large negative curvature at phase zero helps the displacement estimate. Positive curvature can arise only where $F_h' < 0$, and is bounded by

$$p_{\mathrm{w}}|k|^2 \big(-F_h'(k \cdot x)\big)_+ \leq Ch\, p_{\mathrm{w}}|k|^2, \tag{1.11}$$

where $(r)_+ = \max\{r, 0\}$. Indeed $p_{\mathrm{w}}|k|^2 = 2\nu_+ |\mathbf{a}|\,|k|$. At the origin the normal curvature is $-2\nu_+$ times the shear strength, whereas the positive curvature anywhere in this affine wave is at most $2Ch\nu_+|\mathbf{a}|\,|k|$. Over a period the derivative of F_h must have mean zero. The normalized sine profile has slope 1 at zero and -1 at phase $1/2$. We instead concentrate the positive slope in a narrow interval of width $O(h)$. A shallow negative slope over the rest of the period balances its integral (Figure 2). This also gives $\|F_h\|_\infty = O(h)$, so the wave velocity is $O(h|\mathbf{a}|)$.

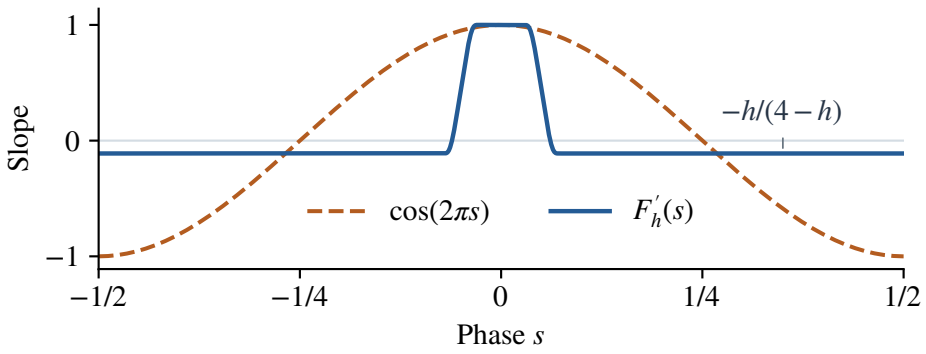


FIGURE 2. Slopes of two profiles over one period. The sine slope $\cos(2\pi s)$ and the modified F_h' both equal one at the center, giving the same central shear. For positive wave pressure coefficient, the shallow negative part of F_h' bounds the positive pressure curvature. The illustrative plateau is $-h/(4-h)$ for a bump of integral $1/4$ in Lemma 3.1. Its plotted height is approximately $-1/9$ ($h \simeq 0.4$).

For the localized wave the pressure coefficient remains positive along the central trajectory throughout amplification. Estimate (5.19) extends this sign to the small envelope, where the spatial cutoff is nonnegative. The unit slope at zero, the small height of the profile and the shallow negative slope give three different powers of the frequency. Normalize the maximum central amplitude on the wave's interval to $\lambda^{-3/4}$, where λ is the length of the normal at launch, and take $h = \lambda^{-1/2}$. These choices give the iteration three simultaneous bounds. The leading velocity is

$O(\lambda^{-5/4})$. At a time when the cutoff is one, the central shear has size

$$|\mathbf{a}(t,0)|\,|\kappa(t,0)| = \lambda^{1/4}\frac{|\mathbf{a}(t,0)|}{\max|\mathbf{a}(\cdot,0)|}\frac{|\kappa(t,0)|}{\lambda},$$

where κ is the transported normal. Proposition 2.8 controls the product of the amplitude ratio and the normal stretch at the successor's launch. Up to a fixed power of the parent bound, the coefficient of $-F_h'$ in the leading pressure curvature is at most $C\lambda^{1/4}$. With the same parent factor the positive curvature is at most $C\lambda^{-1/4}$ (Lemma 5.9). Section 7 sums these upper bounds and estimates $\|\nabla R_U\|_\infty$ to keep $H^\# \le \mathrm{Id}$ at successive steps.

These powers describe this choice of amplitude and profile. The steep part of the profile has phase width $O(h)$. At the origin at launch its physical thickness is of order $h/\lambda = \lambda^{-3/2}$. Relative to the velocity bound, one spatial derivative costs $\lambda^{3/2}$, up to factors from the parent flow. Interpolating the bounds for the velocity and its first derivative gives the power $\lambda^{-5/4+3\alpha/2}$ in C^α. Its exponent is negative precisely for $\alpha < 5/6$. Section 8.4 handles the parent factors and sums the increments for each such α along the same sequence.

1.3. **Correcting a localized wave.** Throughout one correction step we keep the parent U fixed. Its Fourier support is bounded, so Bernstein's inequality controls its higher derivatives. Its residual R_U includes the error from the preceding truncation.

For the flow X of U from time zero, assign each particle label a in the fundamental cell a phase $k_* \cdot a$, with $k_* \neq 0$. Thus

$$\theta(t, X(t,a)) = k_* \cdot a, \qquad \kappa = \nabla_x \theta.$$

The particle keeps its phase, but the phase surfaces deform and their normal κ changes. For a wave after the first, k_* is chosen so that $\kappa = \lambda E_3$ at the origin at launch. To describe the velocity, first keep the phase s independent in a periodic profile $A_1(t,x,s)$ and then put $s = \theta(t,x)$. A spatial derivative of the evaluated profile differentiates both x at fixed s and the phase, with factor κ. A smooth time cutoff $\widetilde{\chi}$ switches on each later wave during the first part of its amplification interval. The first wave is present at time zero and uses $\widetilde{\chi} = 1$.

Write the correction to U as a finite sum of profiles, evaluated at $s = \theta(t,x)$, in the form

$$\sum_{p=1}^{P} Z_p + C_{P+1}, \qquad Z_p = V_p + C_p + B_p.$$

Here B_p is independent of phase, but can depend on time and position. The oscillation V_p has phase average zero and is perpendicular to κ. Differentiation at fixed phase can still give it a divergence. A term C_{p+1} completes it to a divergence-free velocity, as in Figure 3. That term is smaller and is counted at the next level. Initially $V_1 = \widetilde{\chi} A_1$ and $C_1 = B_1 = 0$.

The first error of the expansion, H_2, is produced by the leading wave and C_2. At each (t,x) integrate in s over one period to obtain its phase average $\langle H_2\rangle_s$, leaving an oscillatory part $H_2 - \langle H_2\rangle_s$. Cheverry likewise obtains separate equations for the mean and oscillatory profiles [34, §4.1, (4.19), (4.21)]. Here the mean must also remove the parent residual. With $D_t = \partial_t + U\cdot\nabla_x$, its equation is

$$D_t B_2 + (\nabla U)B_2 + \nabla q = -\langle H_2\rangle_s - R_U, \qquad \operatorname{div} B_2 = 0.$$

The endpoint inverse selects the initial velocity $B_2(0)$ that leaves each particle's final position unchanged to first order. This equation holds on the whole interval of the step. Thus even a wave activated later can require a change in the initial velocity.

With B_2 determined, the oscillatory correction V_2 and its pressure π_2 solve

$$\begin{aligned} D_t V_2 + (\nabla U)V_2 + \kappa\partial_s\pi_2 &= -(H_2 - \langle H_2\rangle_s) - (B_2\cdot\kappa)\partial_s(\widetilde{\chi}A_1), \\ \kappa\cdot V_2 = 0, \qquad \langle V_2\rangle_s &= 0. \end{aligned} \tag{1.12}$$

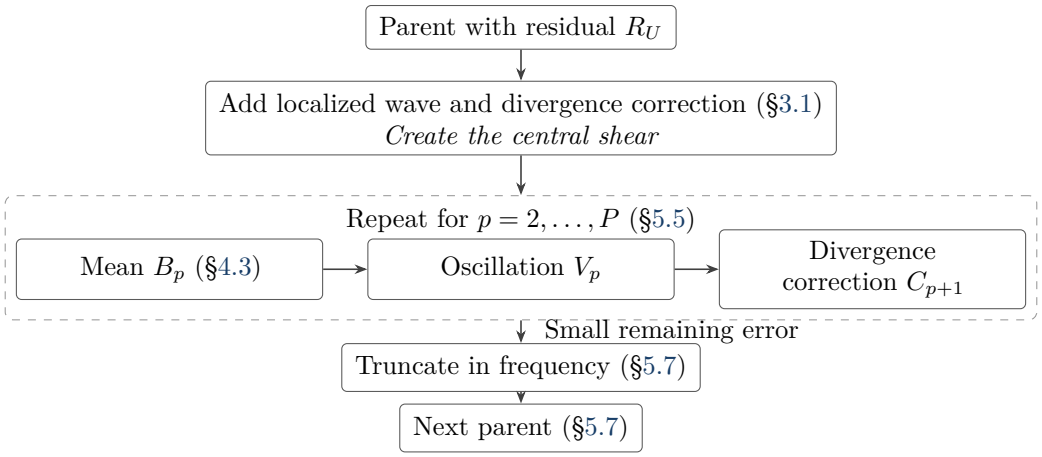


FIGURE 3. Adding and correcting one wave. The mean B_p transports the leading wave and so enters the equation for V_p. The curl correction C_{p+1} completes V_p and enters the next level. After the last level, frequency truncation produces a new residual which must be small at the next wave's scale.

Throughout the step the phase follows U. The additional transport of the leading wave by B_2 therefore appears on the right as $-(B_2 \cdot \kappa)\partial_s(\widetilde{\chi} A_1)$. This term has zero phase average because it differentiates a periodic function in s. At fixed phase along each trajectory, (1.12) is a forced Craik–Criminale amplitude equation. For each label a and phase s, this equation determines $V_2(t, X(t,a), s)$ from zero initial data. The leading amplitude $\mathbf{a}$ starts from the nonzero, tilted direction of Subsection 1.1 for every wave after the first. Perpendicularity to κ cancels the fast part of the divergence, leaving the spatial derivatives at fixed phase. The curl construction of Subsection 3.1 gives the divergence correction C_3 of V_2, so that $V_2 + C_3$ is exactly divergence free. The term C_3 belongs to level three, where we solve for the mean and oscillatory corrections again.

In the whole-space construction of OpenAI [56, §§3.3–3.4, (3.19), (3.27)–(3.28)], a condition on the initial mean velocity makes it compactly supported. For waves activated at $t_0 > 0$, the forced transverse correction there has zero displacement at both ends of $[0, t_0]$. Here transverse corrections start with zero velocity at every step. The stages in the whole-space construction are exact Euler solutions, and the limiting datum has maximal lifespan at most their accumulation time [56, §6.2]. Here the approximate periodic flows converge in every spatial C^m norm uniformly on each earlier interval, and the central vorticity identifies the accumulation time with the maximal lifespan.

At wave j the expansion stops after a finite number P of levels, retaining C_{P+1} to complete the last oscillation. Theorem 5.6 gains $\lambda^{-2/3}$ per level up to parent factors. The residual also contains the derivative of the time cutoff, supported while the wave is exponentially small relative to its peak. Finally the corrected velocity is truncated in Fourier space. The depth, gain and truncation frequency make the combined residual small enough for wave $j+1$ (Proposition 5.11).

The oscillation vanishes up to its launch and is thereafter confined to a shrinking neighborhood of the origin before Fourier truncation. The phase-independent correction and the Fourier tail may already be present at time zero. Figure 4 shows why only the latter terms need to be summed in high spatial norms on any closed interval preceding the accumulation time.

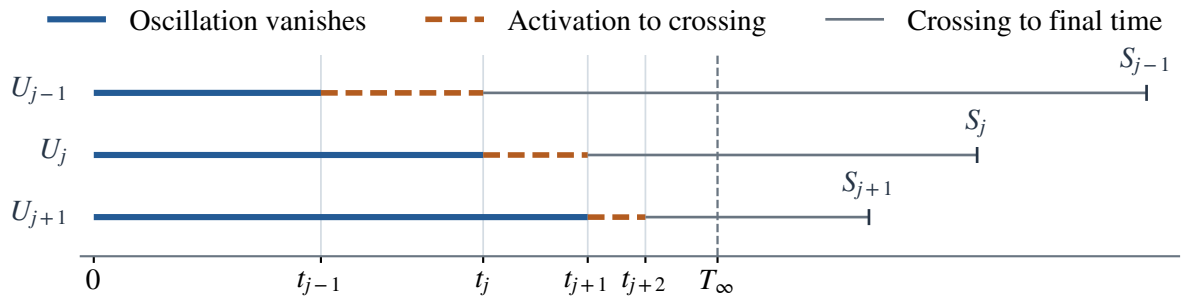


FIGURE 4. Three successive approximate velocities, with $j \geq 4$. The field U_i is defined from time zero to S_i. On the thick segment $[0, t_i]$ the oscillation O_i vanishes. The phase mean and Fourier tail can already be present. On the dashed segment this wave is switched on and reaches the crossing that permits the next launch, at t_{i+1}. Each S_i lies beyond the limit T_∞ of the launch times.

1.4. **Related singularity constructions.** On $\mathbb{R}^3$, Elgindi [15] constructed blow-up from C^{1,α_0} data for some $\alpha_0 > 0$. Elgindi, Ghoul and Masmoudi [16, Theorems 1 and 2] gave finite-energy, locally self-similar examples stable under small axisymmetric perturbations in their weighted Sobolev spaces, including perturbations with swirl. For smooth data, Chen and Hou [28, Theorem 5][29] treat an axially periodic cylinder with an impermeable wall, using computer-assisted estimates. OpenAI [56, Theorem 1.1] obtains finite maximal lifespan from smooth compactly supported data on $\mathbb{R}^3$. The velocity gradient is unbounded along a sequence of times, and the time integral of the vorticity supremum diverges. In the periodic construction here the vorticity tends to infinity at one specified particle throughout the approach to the singular time, while the velocity has a limit on the whole torus.

The forced Boussinesq construction of Alpöge and Buckmaster [1, Section 1.2] uses a force to rotate the temperature gradient and the wave direction together. This preserves their amplifying coupling while changing the horizontal wave number and the sign of vorticity production. A shooting parameter returns the horizontal wave number and the vorticity amplitude to zero. The temperature gradient retains its gain, and a successor grows while the predecessor's wave vector stays vertical.

For forced axisymmetric Euler with swirl [2, Section 1.2], circulation replaces temperature and couples to azimuthal vorticity. Angular control returns the central azimuthal-vorticity amplitude to zero. Radial coefficients require amplitudes that vary across the transported support. In both forced constructions a smooth force also absorbs the final error. The Boussinesq layers even remain exactly linear on a neighborhood of their successors, where the streamfunction corrections vanish. Here incompressibility determines the wave pressure and the wave equations supply the turn. We retain the gradient at the central particle, estimate its variation across the next wave, and send the residual to zero.

The successive scales owe much to Córdoba and Martínez-Zoroa's constructions for forced Euler [18, Section 1.2] and the forced incompressible porous media equation [19, Sections 1.2.1–1.2.4]. With smooth space-time forcing the latter construction is developed in [3, Theorem 2.1]. Córdoba, Martínez-Zoroa and Zheng [17, Theorem 2] used layers of vorticity on $\mathbb{R}^3$ for unforced Euler with $C^{1,\alpha}$ initial data smooth away from the origin. For blow-up with forcing in Boussinesq, variable-density Euler and hypodissipative Navier–Stokes, see [20, Theorem 1], [21, Theorem 1] and [22, Theorem 1].

For the equations themselves see [42], and for the three-dimensional regularity problem see [47]. The contrast with two dimensions is vorticity transport, which gives global solutions for smooth data [45] and even for bounded vorticity [46]. For the three-dimensional local theory see Lichtenstein and Günther [43, 44] and [10]. Numerical studies include the Taylor–Green vortex [48], antiparallel vortex tubes [49, 50] and blow-up at a cylinder wall [51]. Childress, Gilbert and Valiant studied vorticity growth in an eroding dipole [52], and Childress and Gilbert proposed a hairpin geometry for blow-up [53].

Successive frequencies also occur in the construction of weak Euler solutions with nonconstant energy [23, 6]. For nonuniqueness of planar Euler see [24, 25, 26, 27]. In three-dimensional Navier–Stokes the corresponding literature includes conditional nonuniqueness through forward self-similar solutions [54], finite-energy weak solutions [7], Leray–Hopf solutions with a common force [4], and a preprint proof for unforced Leray–Hopf solutions [8].

Reading the proof. Section 2 proves the crossing estimate and Section 3 transports the wave on the torus. Section 4 solves for the initial mean velocity. Section 5 combines that inverse with the transverse equation and estimates the corrected wave. Theorem 7.1 iterates the step, starting from the stationary shear. Section 8 takes the limit, then uses the preceding wave on each launch interval to locate the singularity.

Conventions. Space norms are taken on $\mathbb{T}^3$ unless a subset is specified. A time supremum is indicated by C_t or by the time interval. The Leray projection onto divergence-free vector fields is $\mathbb{P} = I - \nabla\Delta^{-1}\operatorname{div}$, where Δ^{-1} acts on zero-mean scalars. The periodic velocities constructed below are odd and hence have zero spatial mean. We choose zero spatial mean for pressures recovered from a Poisson equation and for the pressure in the endpoint inverse. The assembled pressure of Subsection 5.6 is given by its profile sum. The brackets $\langle f\rangle_s$ indicate integration over one period of the independent phase, at fixed time and position. Spatial differences use the periodic metric, and flows are lifted so that their displacements are periodic. We use $(\nabla U)_{ij} = \partial_j U_i$ and $(\operatorname{div} A)_i = \sum_j \partial_j A_{ij}$. In an orthonormal basis $A_{ij} = E_i \cdot AE_j$. The tensor $a \otimes b = ab^T$ sends a vector v to $(b \cdot v)a$.

1.5. **Choice of parameters.** One sequence must converge through every fixed derivative order. At step j let λ_j denote the wave number at launch, q_j the order to which the error is controlled, and Λ_j the Fourier cutoff after correction. We keep q_j fixed until the preceding frequency can absorb the constants at order $q_j + 1$. The frequencies grow without bound, so every fixed order is eventually reached.

On a first reading the two facts to retain are that $q_j \to \infty$ and that the future logarithmic gain satisfies the bound below. Section 7 uses the full recursion. The frequencies and target ratios are fixed in advance. Each launch after the first two is timed by a crossing of the prescribed ratio.

With the smooth profiles fixed, let b be the exponent in Subsection 5.6 and put $K(s) = 320b(s+1)$. Choose a nondecreasing threshold $\mathcal{B}(s)$ large enough for all the estimates through order and depth $7K(s)$ specified in Appendix C. We also require

$$\log \mathcal{B}(s) \geq 4K(s+1)^3 \qquad (s \geq 1). \tag{1.13}$$

Choose the initial frequency parameter $N \geq \mathcal{B}(1)$ and start at $q_1 = 1$. The recursion reads

$$\begin{gathered} K_j = K(q_j), \qquad \lambda_0 = N, \qquad \lambda_j = \lambda_{j-1}^{K_j} \quad (j \geq 1), \qquad \Lambda_j = \lambda_j^5 \quad (j \geq 0), \\ q_{j+1} = \max\{1 \leq s \leq q_j + 1 : \ \mathcal{B}(s) \leq \lambda_{j-1}\} \quad (j \geq 1), \\ L_j = 4\log\lambda_{j+1}, \qquad \varepsilon_{j-1} = L_j^{-2} \quad (j \geq 2). \end{gathered} \tag{1.14}$$

The stationary shear has cutoff Λ_0. For $j \geq 2$, L_j measures the logarithmic amplification needed to keep the wave small while its time cutoff changes. By (2.24) its amplitude then is at most

$\exp(3 - L_j)$ times its maximum on the comparison interval. The following bound keeps a later logarithmic gain small compared with powers of the frequency already available.

Lemma 1.3. *The integers q_j satisfy $q_j \le q_{j+1} \le q_j + 1$ and tend to infinity, and*

$$L_{j+1} \le (\log \lambda_{j-1})^2 \qquad (j \ge 1). \tag{1.15}$$

For the smooth extension away from the origin, the auxiliary argument [31, Theorem 1.1] includes its mixed-derivative constants in $\mathcal{B}$ before the choice of N. Each fixed order then applies to this same sequence.

2. The wave and its amplification

2.1. **Exact waves over a linear background.** The calculation has three uses. An exact wave in a linear Euler flow gives equations for its normal and amplitude. Those equations still make sense for a prescribed matrix, and we shall solve a strong-shear reference problem in the moving frame of a preceding wave. Finally, a quantitative comparison will transfer its growth, pressure sign and successor geometry to the central gradient of the periodic construction.

Begin with a linear velocity Dx on $\mathbb{R}^3$. If $D(t)$ is smooth and trace free, with $D' + D^2$ symmetric, the background Dx has pressure $-x \cdot (D' + D^2)x/2$. Choose a smooth odd 1-periodic profile F with $F'(0) = 1$. Oddness gives F zero phase mean, so it has a periodic primitive P. Choose its additive constant so that $\langle P \rangle_s = 0$, where $\langle H \rangle_s = \int_0^1 H(s)\, \mathrm{d}s$.

Lemma 2.1. *Let the wave vector $k(t) \neq 0$ and the amplitude vector $\mathbf{a}(t)$, functions of time alone, satisfy*

$$k' = -D^T k, \qquad \mathbf{a}' = -D\mathbf{a} + p_{\mathrm{w}} k, \qquad p_{\mathrm{w}} = \frac{2k \cdot D\mathbf{a}}{|k|^2}, \qquad k \cdot \mathbf{a} = 0. \tag{2.1}$$

Then $Dx + \mathbf{a}F(k \cdot x)$ is divergence free and solves the Euler equations on $\mathbb{R}^3$ with pressure $-x \cdot (D' + D^2)x/2 - p_{\mathrm{w}} P(k \cdot x)$. If $k(t_0) \neq 0$, $\mathbf{a}(t_0) \neq 0$ and $k(t_0) \cdot \mathbf{a}(t_0) = 0$, then both vectors remain nonzero and perpendicular for every $t \ge t_0$ in the interval of existence of D.

Proof. Write $w = \mathbf{a}F(k \cdot x)$ and $u = Dx + w$. Since $k' = -D^T k$,

$$(\partial_t + Dx \cdot \nabla)(k \cdot x) = (k' + D^T k) \cdot x = 0, \qquad (\partial_t + Dx \cdot \nabla) w = \mathbf{a}' F.$$

Also $(w \cdot \nabla) Dx = D\mathbf{a}F$, and perpendicularity cancels the self-advection,

$$(w \cdot \nabla) w = (k \cdot \mathbf{a}) F F' \mathbf{a} = 0.$$

Consequently

$$\partial_t u + (u \cdot \nabla) u = (D' + D^2)x + (\mathbf{a}' + D\mathbf{a})F.$$

The first term is minus the gradient of the background pressure, and the second equals $p_{\mathrm{w}} k F$, which is minus the gradient of the wave pressure $-p_{\mathrm{w}} P(k \cdot x)$. Moreover $\operatorname{div} u = \operatorname{tr} D + (k \cdot \mathbf{a}) F' = 0$. The choice of p_{w} gives $(k \cdot \mathbf{a})' = -2k \cdot D\mathbf{a} + p_{\mathrm{w}} |k|^2 = 0$. For a given D, the equation for k is linear and has an invertible fundamental matrix. Hence $k(t_0) \neq 0$ implies $k(t) \neq 0$. With this k, the equation $\mathbf{a}' = -(\mathrm{Id} - 2\frac{k}{|k|}\frac{k^T}{|k|}) D\mathbf{a}$ is also linear with an invertible fundamental matrix, so $\mathbf{a}(t_0) \neq 0$ implies $\mathbf{a}(t) \neq 0$. □

Transversality has done two jobs in this calculation. It removes self-advection because motion along $\mathbf{a}$ leaves the phase unchanged, and its time derivative determines the pressure coefficient [12, §2, (2.4)–(2.10)]. For the localized wave, the same perpendicularity eliminates the phase derivative in self-advection, and the pressure coefficient is determined along each particle by the same constraint. Spatial variation of the envelope and amplitude can still contribute to self-advection.

2.1.1. *The preceding frame.* A wave evolving over a background matrix B becomes part of the background for the next wave. At the origin, its contribution to the velocity gradient is $\mathbf{a} \otimes k$, and its unit wave vector and velocity direction provide a moving orthonormal frame. We express the older gradient B in this frame to identify the couplings that drive the next wave. The following calculation uses only $\operatorname{tr} B = 0$; it requires no symmetry of $B' + B^2$.

Lemma 2.2. *Let $B(t)$ be a smooth trace-free matrix, and let $k \neq 0$ and $\mathbf{a} \neq 0$ satisfy (2.1) with D replaced by B. Put $E_1 = \frac{k}{|k|}$, $E_2 = \frac{\mathbf{a}}{|\mathbf{a}|}$ and $E_3 = E_1 \times E_2$, write $B_{ij} = E_i \cdot BE_j$ $(i, j = 1, 2, 3)$ for the entries of B in this orthonormal frame, and let $\sigma = |k|\,|\mathbf{a}|$, $\nu = B_{12}$ and $\eta = B_{32}$. Then*

$$E_1' = -\nu E_2 - B_{13}E_3, \quad E_2' = \nu E_1 - \eta E_3, \quad E_3' = B_{13}E_1 + \eta E_2, \qquad (\log \sigma)' = B_{33}. \tag{2.2}$$

Proof. For an arbitrary nonzero vector $v(t)$ depending on time, put $\widehat{v} = v/|v|$. In this proof primes denote derivatives in t. Then

$$\widehat{v}' = (\mathrm{Id} - \widehat{v}\widehat{v}^T)\frac{v'}{|v|}, \qquad (\log |v|)' = \widehat{v} \cdot \frac{v'}{|v|}.$$

Since $B^T E_1 = \sum_j B_{1j} E_j$ and $k' = -B^T k$, this gives

$$E_1' = -(\mathrm{Id} - E_1 E_1^T) B^T E_1 = -B_{12}E_2 - B_{13}E_3, \qquad (\log |k|)' = -B_{11}.$$

Similarly $BE_2 = \sum_i B_{i2} E_i$ and $p_{\mathrm{w}}|k| = 2|\mathbf{a}|B_{12}$, so

$$E_2' = (\mathrm{Id} - E_2 E_2^T)(-BE_2 + 2B_{12}E_1) = B_{12}E_1 - B_{32}E_3, \qquad (\log |\mathbf{a}|)' = -B_{22}.$$

Using $E_3 \times E_2 = -E_1$ and $E_1 \times E_3 = -E_2$ gives

$$E_3' = E_1' \times E_2 + E_1 \times E_2' = B_{13}E_1 + B_{32}E_2.$$

Finally, $(\log \sigma)' = -B_{11} - B_{22} = B_{33}$ because $\operatorname{tr} B = 0$. □

2.2. **Amplification on a prescribed interval.** A large velocity gradient is useful a second time only if the wave which created it can amplify a successor. The frame of the preceding wave distinguishes two entries of the older gradient, $\nu = E_1 \cdot BE_2$ and $\eta = E_3 \cdot BE_2$. The first sets the growth rate together with the shear strength σ. The second rotates the normal out of its launch direction. Their ratio fixes the normalized time available before the reference amplitude peaks. Both couplings must then be estimated in the *new* frame, where their ratio determines the next launch.

We first solve the wave equations with these two entries held constant in the preceding wave's moving frame and pass to the strong-shear limit (Lemma 2.3). We then compare the actual coefficients and amplitudes with this reference system on a prescribed interval. Dividing out the growing reference amplitude in (2.15) lets us control the wave's direction. Near the successor's launch η_+/ν_+ is small. Lemma 2.7 compares it with $\cos\varphi$ to relative accuracy, enough to locate a crossing even when the target ratio tends to zero. Proposition 2.8 bounds the shear at that crossing and the amplitude in the first unit of normalized time. The wave is small while its time cutoff changes, so activation produces little error. At the later crossing the product of its amplitude and its wave number is large, so the next wave can grow.

2.2.1. *The new wave in the preceding frame.* Let $B(t)$ be a smooth trace-free matrix, and let the preceding wave $(k_p, \mathbf{a}_p)$, with $k_p \neq 0$ and $\mathbf{a}_p \neq 0$, satisfy (2.1) with D replaced by B. Let (E_1, E_2, E_3) be its frame, $B_{ij} = E_i \cdot BE_j$ the entries of B in this frame, and $\sigma_p = |k_p|\,|\mathbf{a}_p|$ its strength, as in Lemma 2.2. Thus (2.2) holds with the functions B_{12}, B_{32} and σ_p in the places of ν, η and σ. The new wave evolves over

$$M = B + \mathbf{a}_p \otimes k_p + E_c = B + \sigma_p E_2 \otimes E_1 + E_c,$$

the sum of the older gradient, the gradient of the preceding wave at the origin, and a continuous matrix $E_c(t)$ of errors. Its wave vector k solves the first equation of (2.1) with $D = M$ and equals

λE_3, with $\lambda > 0$, at the activation time t_0. For this k, an *amplitude vector* is a solution $\mathbf{a}$ of the second equation in (2.1) with $D = M$ satisfying $\mathbf{a}(t) \cdot k(t) = 0$. Only these ordinary differential equations enter this subsection: the matrices B and E_c are prescribed, and no equation is assumed for them. Subsection 6.2 applies the results below with M the velocity gradient of the parent at the origin, which is a fixed point of the flow.

Freeze the strength and the two couplings at the activation time,

$$\sigma = \sigma_p(t_0), \qquad \nu = B_{12}(t_0) > 0, \qquad \eta = B_{32}(t_0) > 0,$$

and put

$$r = \sqrt{2\sigma\nu}, \qquad S = \frac{\sigma}{r} = \sqrt{\frac{\sigma}{2\nu}}, \qquad \varepsilon = \frac{\eta}{\nu}, \qquad \tau = r(t - t_0).$$

Since $\nu/\sigma = 1/(2S^2)$, the growth rate $\sqrt{2\nu(\sigma - \nu)}$ found in Subsection 1.1 equals $r(1 - 1/(2S^2))^{1/2}$; thus r is this rate up to a factor close to one when S is large. The variable τ measures time in units of the growth time r^{-1}, and S is the ratio of the preceding shear rate to the growth rate. We work on the interval $0 \le \tau \le T$, or $t_0 \le t \le t_0 + T/r$, with $T \ge 2$. In (2.1) and (2.2) primes denote derivatives in t. From now on in this subsection they denote derivatives in τ, so that $d/d\tau = r^{-1} d/dt$. All suprema are over $0 \le \tau \le T$, and $b_{ij} = B_{ij}/r$.

Define the scaled components $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of the wave vector and an amplitude vector by

$$k = \lambda \left(\zeta_1 E_1 + \frac{\zeta_2}{S} E_2 + \zeta_3 E_3 \right), \qquad \mathbf{a} = \frac{\alpha_1}{S} E_1 + \alpha_2 E_2 + \frac{\alpha_3}{S} E_3. \tag{2.3}$$

Then $\zeta(0) = (0, 0, 1)$ and

$$|k|^2 = \lambda^2 m, \qquad m = \zeta_1^2 + \zeta_2^2/S^2 + \zeta_3^2.$$

Transversality is $\zeta \cdot \alpha = 0$, so where $\zeta_3 \neq 0$,

$$\alpha_3 = -\frac{\zeta_1 \alpha_1 + \zeta_2 \alpha_2}{\zeta_3}.$$

The two components α_1, α_2 therefore determine the amplitude vector. Two small components can contribute at leading order. In the wave pressure, for instance, $\sigma(k \cdot E_2)(\mathbf{a} \cdot E_1) = 2\nu\lambda\zeta_2\alpha_1$, since $\sigma = 2\nu S^2$. The factor S^{-1} in the E_2 coordinate of k makes the contribution of B to the coefficient of ζ_3 in the equation for ζ_2 equal to $-B_{32}/\nu$, which equals $-\varepsilon$ at $\tau = 0$. In the growing mode of Subsection 1.1, the components of the amplitude along E_1 and E_2 have opposite signs, and the first is $S^{-1}(1 - \nu/\sigma)^{-1/2}$ times the second in absolute value. With the scaling in (2.3), this mode is $\alpha_1 = -\alpha_2$ to leading order. In Subsection 2.2.2 the components (α_1, α_2) are divided further by the amplitude in the strong-shear limit, which gives the normalized coordinates (2.15).

In the model case of Subsection 1.1, the entries of B in the frame are the constants $B_{12} = \nu$ and $B_{32} = \eta$, all other entries vanish, and $E_c = 0$. Put

$$\tan\varphi = \tfrac{1}{2}\varepsilon\tau^2, \qquad 0 \le \varphi < \tfrac{\pi}{2}, \qquad \zeta_0 = (\tan\varphi, -\varepsilon\tau, 1). \tag{2.4}$$

As the strength of the preceding shear tends to infinity with ε and T fixed, the new wave vector tends, in the moving frame, to $\lambda(\tan\varphi E_1 + E_3)$. This vector lies in the (E_1, E_3)-plane, at angle φ from E_3 toward E_1. We use the same explicit φ as a reference at finite shear. The wave vector can then have an E_2 component, and its projection can turn through a different angle. Lemma 2.4 bounds the difference in scaled components.

Lemma 2.3. *Suppose that on $0 \le \tau \le T$ the entries of B in the frame are $B_{12} = \nu$, $B_{32} = \eta$ and $B_{ij} = 0$ otherwise, and that $E_c = 0$. Then $\sigma_p = \sigma$, and the scaled components of the wave vector satisfy*

$$\zeta_1' = -\Big(1 + \frac{1}{2S^2}\Big)\zeta_2, \qquad \zeta_2' = -\varepsilon\zeta_3, \qquad \zeta_3' = \frac{\varepsilon}{2S^2}\zeta_2.$$

Where $\zeta_3 \neq 0$, the components α_1, α_2 of every amplitude vector satisfy

$$\alpha_1' = -\alpha_2 + \zeta_1 \Pi,$$
$$\alpha_2' = -\Big(1 - \frac{1}{2S^2}\Big)\alpha_1 + \frac{\varepsilon}{2S^2}\,\frac{\zeta_1\alpha_1 + \zeta_2\alpha_2}{\zeta_3} + \frac{\zeta_2}{S^2}\,\Pi, \qquad \Pi = \frac{2\zeta_2\alpha_1 + (\zeta_1 + \varepsilon\zeta_3)\alpha_2}{m}.$$

As $S \to \infty$ with ε and T fixed, $\zeta \to \zeta_0$ uniformly on $[0,T]$, and the coefficients of this system for (α_1, α_2) converge uniformly on $[0,T]$ to those of

$$\alpha_1' = -\cos^2\varphi(1 - \varepsilon\tan\varphi)\alpha_2 - \varepsilon\tau\sin 2\varphi\,\alpha_1, \qquad \alpha_2' = -\alpha_1. \tag{2.5}$$

Proof. In this proof write $k_i = k \cdot E_i$ and $a_i = \mathbf{a} \cdot E_i$, and let a dot denote the derivative in t. The frame equations (2.2) read $\dot{E}_1 = -\nu E_2$, $\dot{E}_2 = \nu E_1 - \eta E_3$ and $\dot{E}_3 = \eta E_2$, and their last equation gives $d(\log\sigma_p)/dt = B_{33} = 0$, so $\sigma_p = \sigma$. Thus $M = \nu E_1 \otimes E_2 + \eta E_3 \otimes E_2 + \sigma E_2 \otimes E_1$, and $ME_1 = \sigma E_2$, $ME_2 = \nu E_1 + \eta E_3$, $ME_3 = 0$. Since $\dot{k} = -M^T k$, we have $\dot{k}_i = -k \cdot ME_i + k \cdot \dot{E}_i$, hence

$$\dot{k}_1 = -(\sigma + \nu)k_2, \qquad \dot{k}_2 = -2\eta k_3, \qquad \dot{k}_3 = \eta k_2.$$

In the middle equation one term $-\eta k_3$ comes from $-k \cdot ME_2$, the other from the rotation $k \cdot \dot{E}_2$. At finite shear these equations give $k_3 = \lambda\cos(\sqrt{2}\eta(t - t_0))$ and $k_1 = (\sigma+\nu)(\lambda - k_3)/\eta$. The ratio k_1/k_3 gives the projected slope wherever $k_3 \neq 0$. Since $\dot{\mathbf{a}} = -M\mathbf{a} + p_{\mathrm{w}}k$ and $M\mathbf{a} = \nu a_2 E_1 + \sigma a_1 E_2 + \eta a_2 E_3$, also $\dot{a}_i = -E_i \cdot M\mathbf{a} + p_{\mathrm{w}}k_i + \mathbf{a} \cdot \dot{E}_i$, hence

$$\dot{a}_1 = -2\nu a_2 + p_{\mathrm{w}}k_1, \qquad \dot{a}_2 = -(\sigma - \nu)a_1 - \eta a_3 + p_{\mathrm{w}}k_2, \qquad p_{\mathrm{w}} = \frac{2(\nu k_1 a_2 + \sigma k_2 a_1 + \eta k_3 a_2)}{|k|^2}.$$

Substitute (2.3), divide by r to pass to the time τ, and use

$$rS = \sigma, \qquad \frac{\nu}{\sigma} = \frac{1}{2S^2}, \qquad \frac{\nu S}{r} = \frac{1}{2}, \qquad \frac{\eta S}{r} = \frac{\varepsilon}{2}, \qquad \frac{\eta}{\sigma} = \frac{\varepsilon}{2S^2}.$$

The equations for k_i give those for ζ, and $\lambda S p_{\mathrm{w}}/r = \Pi$. The equations for a_1 and a_2 then give those for $\alpha_1 = Sa_1$ and $\alpha_2 = a_2$, after substituting $\alpha_3 = -(\zeta_1\alpha_1 + \zeta_2\alpha_2)/\zeta_3$.

As $S \to \infty$, the coefficients of the system for ζ converge to those of $\zeta_1' = -\zeta_2$, $\zeta_2' = -\varepsilon\zeta_3$, $\zeta_3' = 0$, whose solution with $\zeta(0) = (0,0,1)$ is ζ_0. Both systems are linear with bounded coefficients on $[0,T]$, so $\zeta \to \zeta_0$ uniformly there by Gronwall's inequality. In particular $\zeta_3 \to 1$ and $m \to 1 + \tan^2\varphi = \cos^{-2}\varphi$. In the equation for α_1 the coefficients of α_1 and α_2 therefore converge to

$$\frac{2\zeta_1\zeta_2}{m} \to -2\varepsilon\tau\tan\varphi\cos^2\varphi = -\varepsilon\tau\sin 2\varphi, \qquad -1 + \frac{\zeta_1(\zeta_1 + \varepsilon\zeta_3)}{m} \to -\cos^2\varphi(1 - \varepsilon\tan\varphi),$$

since $(\tan\varphi + \varepsilon)\tan\varphi\cos^2\varphi = \sin^2\varphi + \varepsilon\sin\varphi\cos\varphi$. In the equation for α_2 every term except $-\alpha_1$ has a factor S^{-2} and a bounded coefficient, so the two coefficients converge to -1 and 0. □

We call (2.5) the *reference system*. The vector ζ_0 solves the limiting system $\zeta_1' = -\zeta_2$, $\zeta_2' = -\varepsilon\zeta_3$, $\zeta_3' = 0$ for the wave vector.

The fixed-T strong-shear limit identifies the reference coefficients. In the iteration ε decreases and the comparison interval $[0,T]$ lengthens, both with the wave number. To use the reference solution on these intervals we keep the errors in the coefficients quantitative. Put

$$\delta = \left\| \frac{B_{12}}{\nu} - 1 \right\|_\infty + \left\| \frac{B_{32}}{\eta} - 1 \right\|_\infty + \frac{\|B\|_\infty}{r} + \frac{\|E_c\|_\infty}{\nu}. \tag{2.6}$$

Here the suprema are over the normalized comparison interval and the matrix norms are Euclidean operator norms. The first two terms measure how far the two couplings drift from their launch values. The third is at least $\nu/r = 1/(2S)$, so small δ requires strong shear. Rescaling the E_2 component of k and the E_1, E_3 components of $\mathbf{a}$ by S, and time by r, gives a factor as large as $S/r = 1/(2\nu)$ on entries of E_c (Lemma 2.4).

Fix an absolute constant C_0, sufficiently large for the finitely many estimates of this subsection, and assume

$$C_0\varepsilon \le 1, \qquad C_0 T^{C_0}\delta \le \varepsilon. \tag{2.7}$$

All other constants and exponents C, c in this subsection are absolute and do not depend on C_0; they may change from one estimate to the next. The errors below have the form $CT^C\delta$, and by (2.7) such an error is at most ε times any prescribed negative power of T once C_0 is large.

Since $\delta \ge \nu/r = 1/(2S)$, condition (2.7) gives, for $C_0 \ge 2$ and $T \ge 2$,

$$S \ge \frac{1}{2\delta} \ge \frac{C_0^2 T^{C_0}}{2} \ge T^2, \qquad S^{-2} \le 4\delta^2 \le \delta, \tag{2.8}$$

where the last inequality uses $\delta \le 1/4$.

In the applications in Subsection 6.2 and Section 7, the parameter choice (1.14) prescribes ε, and ε^{-1} and T are bounded by powers of the logarithm of the frequency of the preceding wave, while δ is at most a negative power of that frequency times a power of its logarithm. Item 2 of the proof of Theorem 7.1 verifies (2.7) in this way.

For a general background and $E_c = 0$, each coefficient in the proof of the next lemma is a polynomial in ζ, $1/\zeta_3$, $1/m$ and the quantities

$$b_{11},\ b_{22},\ b_{33},\ b_{13},\ b_{31},\ \frac{b_{21}}{S},\ \frac{b_{12}}{S},\ \frac{b_{23}}{S},\ \frac{b_{32}}{S},\ 2Sb_{12},\ 2Sb_{32},\ \frac{\sigma_p}{\sigma},\ S^{-2}. \tag{2.9}$$

The *reference value* of such an expression is its value at $\zeta = \zeta_0$, $2Sb_{12} = 1$, $2Sb_{32} = \varepsilon$ and $\sigma_p/\sigma = 1$, with all other quantities in (2.9) equal to zero. By Lemma 2.3, it is the limit of the expression in the model case as $S \to \infty$.

Lemma 2.4. *Assume (2.7). The scaled components of the wave vector satisfy*

$$\sup|\zeta - \zeta_0| \le CT^C\delta \le \tfrac{1}{2}, \qquad \sup\left|\frac{|k|^2\cos^2\varphi}{\lambda^2} - 1\right| \le CT^C\delta \le \tfrac{1}{2}. \tag{2.10}$$

In particular

$$\tfrac{1}{2} \le \zeta_3 \le \tfrac{3}{2}, \qquad |\zeta| \le 3T^2. \tag{2.11}$$

The components α_1, α_2 of every amplitude vector satisfy

$$\frac{d}{d\tau}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \left\{-\begin{pmatrix}\varepsilon\tau\sin 2\varphi & \cos^2\varphi(1-\varepsilon\tan\varphi)\\ 1 & 0\end{pmatrix} + \mathcal{E}(\tau)\right\}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \qquad \sup|\mathcal{E}| \le CT^C\delta,$$

where $\mathcal{E}$ is the same for all amplitude vectors.

Proof. The equations (2.1) and (2.2) are in the time t, and every coefficient in them is a rate of change in t. In this proof primes denote derivatives in τ. Since $d/d\tau = r^{-1}d/dt$, each coefficient is divided by r, which turns B_{ij} into b_{ij} and σ_p into $S\sigma_p/\sigma$. By (2.6) and $r/S = 2\nu$,

$$|b_{ij}| \le \delta, \qquad |2Sb_{12} - 1| = \left|\frac{B_{12}}{\nu} - 1\right| \le \delta, \qquad |2Sb_{32} - \varepsilon| = \varepsilon\left|\frac{B_{32}}{\eta} - 1\right| \le \delta.$$

In the time τ the last equation of (2.2) reads $(\log\sigma_p)' = b_{33}$, and $\sigma_p = \sigma$ at $\tau = 0$. Hence

$$\frac{\sigma_p(\tau)}{\sigma} = \exp\left(\int_0^\tau b_{33}(\tau')\,d\tau'\right),$$

and $\sup|\sigma_p/\sigma - 1| \le 2T\delta$, because the integral is at most $T\delta \le 1$ in absolute value.

When $E_c = 0$, the equation for k and (2.2) give

$$\zeta_2' = -b_{22}\zeta_2 - 2Sb_{32}\zeta_3, \qquad \zeta_3' = -b_{33}\zeta_3 - (b_{23} - b_{32})\zeta_2/S,$$
$$\zeta_1' = -b_{11}\zeta_1 - (\sigma_p/\sigma + b_{21}/S + b_{12}/S)\zeta_2 - (b_{31} + b_{13})\zeta_3.$$

In the model case these are the equations of Lemma 2.3. When $E_c \neq 0$, the term $-E_c^T k$ is added to the equation for k, and the frame still evolves over B. In the time τ, the additional coefficient matrix for ζ is

$$-r^{-1}\operatorname{diag}(1,S,1)\,E_c^T\operatorname{diag}(1,S^{-1},1),$$

where E_c^T is the transpose of E_c written in the frame (E_i) of the preceding wave, the matrix with entries $E_j \cdot E_c E_i$. Since $S \geq 1$, its norm is at most $(S/r)|E_c| = |E_c|/(2\nu) \leq \delta$.

Write $\zeta' = (A_0 + A_1)\zeta$, where A_0 is the matrix of the limiting system $\zeta_1' = -\zeta_2$, $\zeta_2' = -\varepsilon\zeta_3$, $\zeta_3' = 0$, and A_1 collects the remaining coefficients. The bounds above give $\sup|A_1| \leq CT\delta$. The vector ζ_0 solves $\zeta_0' = A_0\zeta_0$ with the same initial value as ζ, and $|\zeta_0| \leq 2T^2$. There are only two successive transfers between normal components in the limiting equations, from ζ_3 to ζ_2 and then to ζ_1. Thus $A_0^3 = 0$, and the propagator is $\exp(uA_0) = \mathrm{Id} + uA_0 + u^2A_0^2/2$, of norm at most $2T^2$ for $0 \leq u \leq T$. Therefore

$$\zeta(\tau) - \zeta_0(\tau) = \int_0^\tau e^{(\tau-\tau')A_0}A_1(\tau')\zeta(\tau')\,d\tau',$$

so

$$\sup|\zeta - \zeta_0| \leq CT^C\delta\,\big(1 + \sup|\zeta - \zeta_0|\big).$$

Condition (2.7) gives the first bound of (2.10), hence (2.11) and $m \geq 1/4$. Since $m = \zeta_1^2 + S^{-2}\zeta_2^2 + \zeta_3^2$ and $S^{-2} \leq \delta$ by (2.8), it also gives $|m - \cos^{-2}\varphi| \leq CT^C\delta$. Multiplication by $\cos^2\varphi \leq 1$ proves the second bound.

To express the pressure coefficient in the amplitude equation for $E_c = 0$, put

$$\begin{aligned}
F_1 &= \zeta_1(b_{11} - b_{13}\zeta_1/\zeta_3) + \zeta_2(\sigma_p/\sigma + b_{21}/S - (b_{23}/S)\zeta_1/\zeta_3)\\
&\quad + \zeta_3(b_{31} - b_{33}\zeta_1/\zeta_3),\\
F_2 &= \zeta_1(Sb_{12} - b_{13}\zeta_2/\zeta_3) + \zeta_2(b_{22} - (b_{23}/S)\zeta_2/\zeta_3)\\
&\quad + \zeta_3(Sb_{32} - b_{33}\zeta_2/\zeta_3).
\end{aligned}$$

Then the pressure coefficient of (2.1) satisfies

$$\frac{\lambda S}{r}p_{\mathrm{w}} = \frac{2(F_1\alpha_1 + F_2\alpha_2)}{m}.$$

Substituting for α_3 in the first two amplitude equations gives a system for $(\alpha_1,\alpha_2)^T$ whose coefficient matrix N has entries

$$\begin{aligned}
N_{11} &= -b_{11} + 2b_{13}\zeta_1/\zeta_3 + \boxed{2\zeta_1F_1/m},\\
N_{12} &= \boxed{-2Sb_{12}} + 2b_{13}\zeta_2/\zeta_3 + \boxed{2\zeta_1F_2/m},\\
N_{21} &= \boxed{-\sigma_p/\sigma} - (b_{21} - b_{12})/S + ((b_{23} + b_{32})/S)\zeta_1/\zeta_3 + 2\zeta_2F_1/(S^2m),\\
N_{22} &= -b_{22} + ((b_{23} + b_{32})/S)\zeta_2/\zeta_3 + 2\zeta_2F_2/(S^2m).
\end{aligned}$$

Only the boxed terms have nonzero reference values. Every other term has one of b_{11}, b_{22}, b_{13}, the quotients b_{ij}/S or S^{-2} as a factor, so its reference value is zero. In F_1 only the term $\zeta_2\sigma_p/\sigma$, and in F_2 only ζ_1Sb_{12} and ζ_3Sb_{32}, have nonzero reference values. Hence the reference values of F_1, F_2 and m are $-\varepsilon\tau$, $(\tan\varphi + \varepsilon)/2$ and $\cos^{-2}\varphi$. Thus the reference entries of N are

$$-\varepsilon\tau\sin 2\varphi, \qquad -1 + \tan\varphi(\tan\varphi + \varepsilon)\cos^2\varphi = -\cos^2\varphi(1 - \varepsilon\tan\varphi), \qquad -1, \qquad 0,$$

which are the coefficients of (2.5).

Each entry of N is a polynomial of bounded degree in ζ_1, ζ_2, ζ_3, $1/\zeta_3$, $1/m$ and the quantities (2.9). These are at most CT^2. Each differs from its reference value by at most $CT^C\delta$: the coordinates by (2.10), the reciprocals by a multiple of $|m - \cos^{-2}\varphi| + |\zeta_3 - 1|$, the coefficients by the bounds at the beginning of the proof, and S^{-2} by (2.8). Hence each entry of N differs from its reference value by at most $CT^C\delta$.

For $E_c \neq 0$, the amplitude equation contains the additional term $-(\mathrm{Id} - 2\frac{k}{|k|} \otimes \frac{k}{|k|})E_c\mathbf{a}$, of norm at most $|E_c|\,|\mathbf{a}|$. By (2.3), (2.11) and (2.8), $|\mathbf{a}| \le C|(\alpha_1, \alpha_2)|$, while $\alpha_1 = S\mathbf{a} \cdot E_1$ and $\alpha_2 = \mathbf{a} \cdot E_2$. In the time τ this adds a coefficient matrix of norm at most $C(S/r)|E_c| \le C\delta$ to the reduced amplitude system. □

2.2.2. *The reference solution.* At $\tau = 0$, the reference system (2.5) is $\alpha_1' = -\alpha_2$, $\alpha_2' = -\alpha_1$. The growing eigenvector of this frozen launch matrix has $\alpha_1 = -\alpha_2$, the direction $-E_1/S + E_2$ of the amplitude in (2.3). As φ increases, the pressure reduces the coefficient driving α_1 from one to $\cos^2\varphi(1 - \varepsilon\tan\varphi)$ and introduces the term $-\varepsilon\tau\sin 2\varphi\,\alpha_1$. The driving coefficient becomes negative when $\varepsilon\tan\varphi > 1$.

The *reference solution* is the solution of the reference system with $(\alpha_1, \alpha_2)(0) = (-1, 1)$, shown in Figure 1. A later wave is started with these data in Subsection 6.2. The second equation of (2.5), $\alpha_2' = -\alpha_1$, holds at every τ. Hence, wherever $\alpha_2 > 0$, the logarithmic derivative of α_2 is

$$Y = \frac{\alpha_2'}{\alpha_2} = -\frac{\alpha_1}{\alpha_2},$$

and the first equation of (2.5) gives the Riccati equation

$$Y' = \cos^2\varphi(1 - \varepsilon\tan\varphi) - \varepsilon\tau\sin 2\varphi\, Y - Y^2, \qquad Y(0) = 1. \tag{2.12}$$

Conversely, if Y solves (2.12), then $(-Y a_{\mathrm{ref}}, a_{\mathrm{ref}})$ is the reference solution, where

$$a_{\mathrm{ref}}(\tau) = \exp\left(\int_0^\tau Y(\tau')\,d\tau'\right).$$

We call a_{ref} the *reference amplitude*.

Lemma 2.5. *Let $0 < \varepsilon \le 1$. The solution Y of (2.12) exists for all $\tau \ge 0$, satisfies $|Y| \le 1$, and has the form*

$$Y = (1 + e)\cos\varphi - \tfrac{1}{4}\varepsilon\tau\sin 2\varphi, \qquad 0 \le e \le \tfrac{1}{8}\varepsilon. \tag{2.13}$$

The reference amplitude satisfies

$$a_{\mathrm{ref}}(\tau) = \sqrt{\cos\varphi(\tau)}\exp\left(\int_0^\tau (1 + e)\cos\varphi\,d\tau'\right). \tag{2.14}$$

The function Y has exactly one zero τ_m, with $2 \le \varepsilon\tau_m \le 3$. It is positive before this zero and negative afterwards, so a_{ref} is largest at τ_m, and

$$\log a_{\mathrm{ref}}(\tau_m) \ge \varepsilon^{-1/2} - 1.$$

Proof. Since $\tan\varphi = \varepsilon\tau^2/2$,

$$\varphi' = \varepsilon\tau\cos^2\varphi, \qquad (\log\cos\varphi)' = -\tfrac{1}{2}\varepsilon\tau\sin 2\varphi.$$

We look for Y in the form (2.13), an ansatz with an unknown function e. To motivate it, write $Y = \cos\varphi + w$, where $\cos\varphi$ is the positive equilibrium of the equation $Y' = \cos^2\varphi - Y^2$ with φ frozen. Neglecting w', w^2, $\varepsilon\tau w\sin 2\varphi$ and the term $-\varepsilon\sin\varphi\cos\varphi$ of order ε, (2.12) reduces to $2w\cos\varphi = -\varepsilon\tau\sin\varphi\cos^2\varphi$, that is, $w = -\frac{1}{4}\varepsilon\tau\sin 2\varphi = \frac{1}{2}(\log\cos\varphi)'$. The factor $1 + e$ absorbs the neglected terms.

Integrating (2.13) and using the formula for $(\log\cos\varphi)'$ gives (2.14). Substituting (2.13) into (2.12) gives

$$e' = \cos\varphi\big(-2e - e^2 + \tfrac{1}{2}\varepsilon\sin\varphi\cos\varphi\big), \qquad e(0) = 0.$$

The right-hand side is nonnegative at $e = 0$ and nonpositive at $e = \varepsilon/8$, because $\sin\varphi\cos\varphi \le 1/2$. Hence e remains in $[0, \varepsilon/8]$ and exists for all $\tau \ge 0$. This constructs the global solution Y.

Wherever $Y \geq 0$, (2.12) gives $Y' \leq 1 - Y^2$, because $\cos^2\varphi(1 - \varepsilon\tan\varphi) \leq 1$. Since $Y(0) = 1$, $Y(\tau)$ never exceeds one. Also $\sin\varphi\cos\varphi \leq \min\{1/2, \cot\varphi\}$, so for $\tau > 0$,

$$\tfrac{1}{4}\varepsilon\tau\sin 2\varphi \leq \min\{\varepsilon\tau/4, 1/\tau\} \leq \tfrac{1}{2}.$$

Equation (2.13) therefore gives $Y \geq -1/2$.

Write $Y = \cos\varphi(1 + e - \frac{1}{2}\varepsilon\tau\sin\varphi)$. It is positive for $\varepsilon\tau \leq 2$, because $e \geq 0$ and $\sin\varphi < 1$. For $\varepsilon\tau = 3$, we have $\tan\varphi = 9/(2\varepsilon) \geq 9/2$, hence $\sin\varphi > 3/4$ and $\frac{1}{2}\varepsilon\tau\sin\varphi > 9/8 \geq 1 + e$. Thus Y is negative there. At any zero with $\varepsilon\tau \geq 2$,

$$Y' = \cos^2\varphi(1 - \varepsilon\tan\varphi) = \cos^2\varphi\big(1 - (\varepsilon\tau)^2/2\big) < 0.$$

Consequently Y has exactly one zero τ_m, with $2 \leq \varepsilon\tau_m \leq 3$, and has the stated signs on its two sides.

For $0 \leq \tau \leq (2/\varepsilon)^{1/2}$, we have $\tan\varphi \leq 1$, hence $\cos\varphi \geq 2^{-1/2}$. This interval ends before $\tau_m \geq 2/\varepsilon$, where a_{ref} is largest. By (2.14) and $e \geq 0$,

$$\log a_{\mathrm{ref}}(\tau_m) \geq \log a_{\mathrm{ref}}\big((2/\varepsilon)^{1/2}\big) \geq 2^{-1/2}(2/\varepsilon)^{1/2} - \tfrac{1}{4}\log 2 \geq \varepsilon^{-1/2} - 1. \qquad \square$$

An absolute error in amplitude can grow as rapidly as the wave itself. To control its direction over the whole interval, divide out the reference growth and use the reference solution as one coordinate direction. Its second component a_{ref} is positive throughout the interval. Write $\alpha_2 = a_{\mathrm{ref}}h$ and $\alpha_1 = -a_{\mathrm{ref}}(Yh + l\cos\varphi)$. The term in h follows the reference direction, and the term in l measures the departure from it. Here h denotes the relative amplitude. The profile width of Section 3 enters only after the wave comparison. Inverting gives

$$h = \frac{\alpha_2}{a_{\mathrm{ref}}}, \qquad l = -\frac{\alpha_1 + Y\alpha_2}{a_{\mathrm{ref}}\cos\varphi} \tag{2.15}$$

for the *normalized coordinates* of (α_1, α_2). These are linear in the two components even when $\alpha_2 = 0$. The factor $1/\cos\varphi$ in l gives the equations for h and l below the same factor $\cos\varphi$, which yields a propagator bound uniform in ε and in the length of the interval. By the second equation of (2.5) and $a'_{\mathrm{ref}} = Ya_{\mathrm{ref}}$, we have $h' = l\cos\varphi$. The trace of the reference coefficient matrix is $-\varepsilon\tau\sin 2\varphi = 2(\log\cos\varphi)'$. By Liouville's formula, the determinant

$$\det\begin{pmatrix}\alpha_1 & -Ya_{\mathrm{ref}}\\ \alpha_2 & a_{\mathrm{ref}}\end{pmatrix} = a_{\mathrm{ref}}(\alpha_1 + Y\alpha_2) = -a_{\mathrm{ref}}^2 l\cos\varphi$$

is a constant multiple of $\cos^2\varphi$. Formula (2.14) thus gives

$$l(\tau) = l(0)\exp\left(-2\int_0^\tau (1+e)\cos\varphi\, d\tau'\right).$$

The reference system becomes

$$h' = l\cos\varphi, \qquad l' = -2(1+e)l\cos\varphi, \tag{2.16}$$

and the reference solution is $(h, l) = (1, 0)$. The second equation shows that $|l|$ does not increase. Since $e \geq 0$, for $\tau_1 \leq \tau_2$,

$$|h(\tau_2) - h(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |l|\cos\varphi \leq \int_{\tau_1}^{\tau_2} (1+e)|l|\cos\varphi = \tfrac{1}{2}\big(|l(\tau_1)| - |l(\tau_2)|\big).$$

The triangle inequality and $|l(\tau_2)| \leq |l(\tau_1)|$ give

$$|h(\tau_2)| + |l(\tau_2)| \leq |h(\tau_1)| + |l(\tau_1)|.$$

Since $|(h,l)| \leq |h| + |l| \leq \sqrt{2}\,|(h,l)|$, the propagator $\Psi_0(\tau_2, \tau_1)$ of (2.16) has norm at most 2 for $\tau_1 \leq \tau_2$, uniformly in both times and in ε.

2.2.3. *Amplitude and direction relative to the reference solution.* For the perturbed amplitude equation we keep the same coordinates. Dividing by a_{ref} removes the large gain before we estimate the error. The uniform bound just proved for the reference propagator then gives a polynomial loss in T, which (2.7) can absorb. For the selected launch direction the ratio $|\mathbf{a}(\tau)|/a_{\mathrm{ref}}(\tau)$ stays between $\exp(-\varepsilon)$ and $\exp(\varepsilon)$ for every $0 \le \tau \le T$. We also need to propagate arbitrary transverse vectors when the wave is localized. Converting their initial values to the normalized coordinates costs a factor S.

Lemma 2.6. *Assume (2.7), and let* $\mathbf{a}$ *be the amplitude vector with* $\mathbf{a}(t_0) = -E_1(t_0)/S + E_2(t_0)$, *that is, with* $(\alpha_1, \alpha_2)(0) = (-1, 1)$ *in (2.3), the initial data of the reference solution. Then* $\alpha_2 > 0$ *on* $0 \le \tau \le T$, *and*

$$\left| \frac{\alpha_1}{\alpha_2} + Y \right| \le CT^C\delta, \qquad \alpha_2 \le |\mathbf{a}| \le (1 + CT^C\delta)\alpha_2. \tag{2.17}$$

Moreover,

$$\left| \log \frac{|\mathbf{a}|}{a_{\mathrm{ref}}} \right| \le \varepsilon. \tag{2.18}$$

For $0 \le \tau_1 \le \tau_2 \le T$, *let* $\Phi(\tau_2, \tau_1)$ *map* $\mathbf{b}(\tau_1)$ *to* $\mathbf{b}(\tau_2)$ *for every amplitude vector* $\mathbf{b}$. *It maps the plane perpendicular to* $k(\tau_1)$ *onto the plane perpendicular to* $k(\tau_2)$, *and*

$$\|\Phi(\tau_2, \tau_1)\| \le CS \, \frac{|\mathbf{a}(\tau_2)|}{|\mathbf{a}(\tau_1)|}.$$

Proof. For an amplitude vector $\mathbf{b}$, let $\beta = (\beta_1, \beta_2, \beta_3)$ be its scaled components, defined as those of $\mathbf{a}$ in (2.3):

$$\mathbf{b} = \frac{\beta_1}{S} E_1 + \beta_2 E_2 + \frac{\beta_3}{S} E_3.$$

Its normalized coordinates are given by (2.15) with (β_1, β_2) in place of (α_1, α_2), that is,

$$\begin{pmatrix} h \\ l \end{pmatrix} = \frac{1}{a_{\mathrm{ref}}} R \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 \\ -(\cos\varphi)^{-1} & -Y(\cos\varphi)^{-1} \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} -Y & -\cos\varphi \\ 1 & 0 \end{pmatrix}.$$

By Lemma 2.4, (h, l) solves (2.16) with the additional coefficient matrix $R\mathcal{E}R^{-1}$. Since $|Y| \le 1$ and $(\cos\varphi)^{-1} \le T^2$, we have $|R| \le CT^2$, $|R^{-1}| \le C$ and $|R\mathcal{E}R^{-1}| \le CT^C\delta$.

Let $\Psi(\tau_2, \tau_1)$ be the propagator of this perturbed system. Variation of constants gives

$$\Psi(\tau_2, \tau_1) = \Psi_0(\tau_2, \tau_1) + \int_{\tau_1}^{\tau_2} \Psi_0(\tau_2, \tau')(R\mathcal{E}R^{-1})(\tau')\Psi(\tau', \tau_1)\, d\tau'.$$

Consequently $\sup|\Psi| \le 2 + 2T \cdot CT^C\delta \sup|\Psi|$, and (2.7) gives $|\Psi(\tau_2, \tau_1)| \le 4$.

For the prescribed initial amplitude $\mathbf{a}(t_0)$ the normalized coordinates at $\tau = 0$ are $(h, l)(0) = (1, 0)$, a constant solution of the reference system. Applying the preceding formula to this vector gives

$$\sup|(h - 1, l)| \le 2T \cdot CT^C\delta \cdot 4 \le CT^C\delta \le \tfrac{1}{2}.$$

In particular $h \ge 1/2$, and the coordinate identities give

$$\alpha_2 = a_{\mathrm{ref}} h > 0, \qquad \frac{\alpha_1}{\alpha_2} + Y = -\frac{l\cos\varphi}{h}, \qquad \left| \frac{\alpha_1}{\alpha_2} + Y \right| \le 2|l| \le CT^C\delta.$$

This is the first bound of (2.17), and $|Y| \le 1$ gives $|\alpha_1| \le 2\alpha_2$. Since $\alpha_3 = -(\zeta_1\alpha_1 + \zeta_2\alpha_2)/\zeta_3$, the bounds (2.11) give $|\alpha_3| \le 18T^2\alpha_2$. By (2.3),

$$|\mathbf{a}|^2 = \alpha_2^2 + S^{-2}(\alpha_1^2 + \alpha_3^2).$$

Since $S^{-2} \le \delta$ by (2.8), this proves the second bound of (2.17). Thus $\log(|\mathbf{a}|/a_{\mathrm{ref}}) = \log h + \log(|\mathbf{a}|/\alpha_2)$ is at most $CT^C\delta$ in absolute value, and (2.7) gives (2.18).

To bound $\Phi(\tau_2,\tau_1)$, note that for an arbitrary amplitude vector $\mathbf{b}$, $\beta_1 = -a_{\mathrm{ref}}(Yh + l\cos\varphi)$, $\beta_2 = a_{\mathrm{ref}}h$ and $\beta_3 = -(\zeta_1\beta_1 + \zeta_2\beta_2)/\zeta_3$. Hence $\mathbf{b}/a_{\mathrm{ref}} = hL_h + lL_l$, where

$$L_h = -\frac{Y}{S}E_1 + E_2 + \frac{\zeta_1 Y - \zeta_2}{S\zeta_3}E_3, \qquad L_l = \frac{\cos\varphi}{S}\left(-E_1 + \frac{\zeta_1}{\zeta_3}E_3\right).$$

By (2.11) and $S \geq T^2$, $|L_h| + |L_l| \leq C$. Conversely, $h = (\mathbf{b}/a_{\mathrm{ref}})\cdot E_2$, and $lL_l = \mathbf{b}/a_{\mathrm{ref}} - hL_h$. By (2.10),

$$|L_l| = \frac{\cos\varphi}{S}\sqrt{1+\zeta_1^2/\zeta_3^2} \geq \frac{c}{S}, \qquad |h| + |l| \leq CS|\mathbf{b}|/a_{\mathrm{ref}}.$$

Together with $|\Psi| \leq 4$, these two coordinate bounds give

$$\|\Phi(\tau_2,\tau_1)\| \leq CS\,\frac{a_{\mathrm{ref}}(\tau_2)}{a_{\mathrm{ref}}(\tau_1)} \leq CS\,\frac{|\mathbf{a}(\tau_2)|}{|\mathbf{a}(\tau_1)|},$$

where the last step uses (2.18). □

2.2.4. *The couplings of the new wave.* Recall that the new wave evolves over the gradient $M = B + \sigma_p E_2 \otimes E_1 + E_c$ of Subsection 2.2.1. The couplings ν_+, η_+ of M in the frame of the new wave drive its successor. The first controls the sign of the pressure coefficient, since

$$p_{\mathrm{w}} = \frac{2k\cdot M\mathbf{a}}{|k|^2} = 2\nu_+\frac{|\mathbf{a}|}{|k|}.$$

Already at launch the small tilt of $\mathbf{a}$ affects η_+. In the two-entry model with $E_c = 0$, let $n = (1+S^{-2})^{1/2}$. Then $E_{1,+} = E_3$, $E_{2,+} = (E_2 - S^{-1}E_1)/n$ and $E_{3,+} = -(E_1 + S^{-1}E_2)/n$. Taking components of $ME_{2,+}$ gives $\nu_+ = \eta/n$ and $\eta_+ = (-\nu + \sigma/S^2)/n^2 = \nu/n^2$. Dropping the tilt before multiplying by M would reverse the latter sign.

The following lower bound for ν_+ holds throughout the comparison interval at the central particle. Estimate (5.19) extends the sign to the wave's spatial support, where Lemma 5.9 uses it to control positive pressure curvature.

Lemma 2.7. *Assume* (2.7)*, and let* $\mathbf{a}$ *be the amplitude vector of Lemma 2.6. Put* $E_{1,+} = k/|k|$, $E_{2,+} = \mathbf{a}/|\mathbf{a}|$ *and* $E_{3,+} = E_{1,+}\times E_{2,+}$*, and define*

$$\nu_+ = E_{1,+}\cdot ME_{2,+}, \qquad \eta_+ = E_{3,+}\cdot ME_{2,+}.$$

Then $\nu_+ \geq \varepsilon\nu/2$ *on* $0 \leq \tau \leq T$*. Where* $\varepsilon\tan\varphi \geq 1$*, that is, for* $\sqrt{2}/\varepsilon \leq \tau \leq T$*,*

$$\left|\frac{\nu_+}{\nu} - 1\right| \leq \varepsilon, \qquad \left|\frac{\eta_+/\nu_+}{\cos\varphi} - 1\right| \leq \frac{1}{2}. \tag{2.19}$$

Proof. Put

$$q = \frac{\alpha_1}{\alpha_2}, \qquad z = \frac{\alpha_3}{\alpha_2} = -\frac{\zeta_1 q + \zeta_2}{\zeta_3}, \qquad n = \frac{|\mathbf{a}|}{\alpha_2}.$$

Then

$$\frac{\mathbf{a}}{\alpha_2} = \frac{q}{S}E_1 + E_2 + \frac{z}{S}E_3, \qquad n^2 = 1 + \frac{q^2+z^2}{S^2}.$$

For $E_c = 0$, the components of $M\mathbf{a}/(r\alpha_2)$ in the frame (E_1,E_2,E_3) of the preceding wave are $(f_1/S, f_2, f_3/S)$, where

$$f_1 = b_{11}q + Sb_{12} + b_{13}z, \qquad f_2 = (\sigma_p/\sigma + b_{21}/S)q + b_{22} + (b_{23}/S)z, \qquad f_3 = b_{31}q + Sb_{32} + b_{33}z.$$

In the same frame, the components of $(k/\lambda)\times(\mathbf{a}/\alpha_2)$ are

$$\left(\frac{\zeta_2 z}{S^2} - \zeta_3, \frac{\zeta_3 q - \zeta_1 z}{S}, \zeta_1 - \frac{\zeta_2 q}{S^2}\right).$$

Since $E_{1,+} = k/|k|$, $E_{2,+} = \mathbf{a}/|\mathbf{a}|$ and $E_{3,+} = (k \times \mathbf{a})/(|k||\mathbf{a}|)$,

$$\nu_+ = \frac{k \cdot M\mathbf{a}}{|k||\mathbf{a}|}, \qquad \eta_+ = \frac{(k \times \mathbf{a}) \cdot M\mathbf{a}}{|k||\mathbf{a}|^2}.$$

We compute these scalar products in the frame of the preceding wave, pairing k/λ and $(k/\lambda)\times(\mathbf{a}/\alpha_2)$ with $M\mathbf{a}/(r\alpha_2)$. With $|k| = \lambda\sqrt{m}$, $|\mathbf{a}| = n\alpha_2$ and $r/S = 2\nu$ this gives

$$\nu_+/\nu = \frac{2(\zeta_1 f_1 + \zeta_2 f_2 + \zeta_3 f_3)}{\sqrt{m}\, n}, \tag{2.20}$$

$$\eta_+/\nu = \frac{2}{\sqrt{m}\, n^2} \left\{ \left(-\zeta_3 + \frac{\zeta_2 z}{S^2} \right) f_1 + (\zeta_3 q - \zeta_1 z) f_2 + \left(\zeta_1 - \frac{\zeta_2 q}{S^2} \right) f_3 \right\}. \tag{2.21}$$

The matrix E_c adds $E_{1,+} \cdot E_c E_{2,+}/\nu$ to ν_+/ν and $E_{3,+} \cdot E_c E_{2,+}/\nu$ to η_+/ν. Each is at most δ in absolute value.

At the reference values,

$$\zeta = (\tan\varphi, -\varepsilon\tau, 1), \qquad q = -Y, \qquad z = Y\tan\varphi + \varepsilon\tau,$$

and $f_1 = 1/2$, $f_2 = -Y$, $f_3 = \varepsilon/2$, $n = 1$ and $m = \cos^{-2}\varphi$. The two expressions in (2.20)–(2.21) are therefore, respectively,

$$\sin\varphi + \varepsilon\cos\varphi + 2\varepsilon\tau Y\cos\varphi, \qquad (\varepsilon\tan\varphi - 1)\cos\varphi + \frac{2Y^2}{\cos\varphi} + 2\varepsilon\tau Y\sin\varphi.$$

In the second expression the terms $\varepsilon\tan\varphi\cos\varphi$, $2Y^2/\cos\varphi$ and $2\varepsilon\tau Y\sin\varphi$ are large compared with $\cos\varphi$ when $\varepsilon\tan\varphi$ is large, and $\varepsilon\tan\varphi = \frac{1}{2}(\varepsilon\tau)^2$ grows with τ. Their large parts cancel. To see this, write $Y = (1 + e - u)\cos\varphi$ by (2.13), with $u = \frac{1}{2}\varepsilon\tau\sin\varphi$, so that $2\varepsilon\tau\sin\varphi = 4u$ and $2u^2 = \varepsilon\tan\varphi\sin^2\varphi$. Dividing the second expression by $\cos\varphi$ gives

$$\begin{aligned}
&(\varepsilon\tan\varphi - 1) + 2(1 + e - u)^2 + 4u(1 + e - u) \\
&\quad = \varepsilon\tan\varphi - 1 + \big[2(1+e)^2 - 4u(1+e) + 2u^2\big] + \big[4u(1+e) - 4u^2\big] \\
&\quad = 1 + 4e + 2e^2 + \big(\varepsilon\tan\varphi - 2u^2\big).
\end{aligned}$$

The terms $\pm 4u(1+e)$ cancel, and the large parts of $\varepsilon\tan\varphi$ and $2u^2$ cancel:

$$\varepsilon\tan\varphi - 2u^2 = \varepsilon\tan\varphi\,(1 - \sin^2\varphi) = \varepsilon\sin\varphi\cos\varphi \le \tfrac{1}{2}\varepsilon.$$

Similarly, $2\varepsilon\tau Y\cos\varphi = 2(1+e)\varepsilon\tau\cos^2\varphi - 2\varepsilon\sin^2\varphi\cos\varphi$. Thus the reference values of ν_+/ν and η_+/ν are

$$\begin{gathered}
\sin\varphi + \varepsilon\cos\varphi\cos 2\varphi + 2(1+e)\varepsilon\tau\cos^2\varphi, \\
(1 + 4e + 2e^2 + \varepsilon\sin\varphi\cos\varphi)\cos\varphi.
\end{gathered} \tag{2.22}$$

Where $\varepsilon\tan\varphi \ge 1$, we have $\cos\varphi \le \cot\varphi \le \varepsilon$, so $1 - \varepsilon^2 \le \sin\varphi \le 1$. Also $0 \le e \le \varepsilon/8$ and

$$\varepsilon\tau\cos^2\varphi \le \varepsilon\tau\cot^2\varphi = \frac{4}{\varepsilon\tau^3} \le \sqrt{2}\,\varepsilon^2.$$

The two reference values in (2.22) are $1 + O(\varepsilon^2)$ and $(1 + O(\varepsilon))\cos\varphi$.

Where $\varepsilon\tan\varphi \le 1$, we have $\varepsilon\tau \le \sqrt{2}$, hence $Y > 0$ by Lemma 2.5. The unsimplified reference value of ν_+/ν is at least $\sin\varphi + \varepsilon\cos\varphi \ge \varepsilon$.

For the wave, in (2.20)–(2.21), the coefficients differ from their reference values by at most $CT\delta$; ζ differs from ζ_0 by at most $CT^C\delta$; the terms with S^{-2} in those formulas, in m and in n, are at most $CT^C\delta$; and q differs from $-Y$ by at most $CT^C\delta$, by (2.17). The formula for z and the bound $\zeta_3 \ge 1/2$ give the same bound for its change. The numerators are polynomials of bounded degree in these quantities and $1/\zeta_3$, which are at most CT^2, while the denominators $\sqrt{m}\,n$ and $\sqrt{m}\,n^2$

are at least $1/2$. Together with the contribution of E_c, the two ratios therefore differ from their reference values by at most $CT^C\delta$, which is at most $\varepsilon T^{-2}/8$ by (2.7).

In both ranges the reference value of ν_+/ν is at least ε; in the late range this uses $C_0\varepsilon \le 1$. Hence $\nu_+/\nu \ge \varepsilon/2$ on the whole interval. In the late range, $\cos\varphi \ge T^{-2}$. Combining the perturbation bounds with (2.22) gives

$$\left|\frac{\nu_+}{\nu} - 1\right| \le \varepsilon, \qquad \frac{\eta_+}{\nu_+} = (1 + O(\varepsilon))\cos\varphi.$$

The absolute perturbation must be small even compared with the late coupling $\cos\varphi$, which tends to zero as the normal turns. Its lower bound T^{-2} is exactly what permits division by it here. Taking C_0 sufficiently large gives (2.19). □

2.2.5. *Successor crossing and retained shear.* The next wave needs a certain ratio ε_+ of its two couplings. We call any time at which $\eta_+/\nu_+ = \varepsilon_+$ a *crossing*. At the peak τ_m of the reference amplitude, $\cos\varphi$ is comparable to ε. Later it is comparable to $(\varepsilon\tau^2)^{-1}$. By (2.19) the actual ratio can therefore be put on opposite sides of its target at two times after launch. Continuity will provide a crossing at which to start the successor. The other quantity needed there is the actual shear $\sigma_+ = |\mathbf{a}||k|$.

Proposition 2.8. *Assume (2.7), let* $\mathbf{a}$ *be the amplitude vector of Lemma 2.6, with the couplings of Lemma 2.7, and suppose that* $0 < \varepsilon_+ \le \varepsilon/32$ *and* $T \ge 2(\varepsilon\varepsilon_+)^{-1/2}$. *Then the interval* $\tau_m \le \tau \le 2(\varepsilon\varepsilon_+)^{-1/2}$, *where* $\tau_m \ge 2/\varepsilon$ *is the zero of* Y *(Lemma 2.5), contains a crossing. At every crossing in this interval the strength* $\sigma_+ = |\mathbf{a}(\tau)||k(\tau)|$ *satisfies*

$$C^{-1}(\varepsilon\varepsilon_+)^{-1/2} \le \frac{\sigma_+}{\lambda \max_{[0,T]} |\mathbf{a}|} \le C(\varepsilon\varepsilon_+)^{-1/2}. \tag{2.23}$$

Moreover

$$|\mathbf{a}(\tau)| \le \exp\bigl(3 - \varepsilon^{-1/2}\bigr) \max_{[0,T]} |\mathbf{a}| \qquad (0 \le \tau \le 1). \tag{2.24}$$

Proof. By Lemma 2.5, $2/\varepsilon \le \tau_m \le 3/\varepsilon$. Since $\varepsilon_+ \le \varepsilon/32$, we have $(\varepsilon\varepsilon_+)^{-1/2} \ge 4/\varepsilon$. Thus $\tau_m < 2(\varepsilon\varepsilon_+)^{-1/2} \le T$, and $[\tau_m, T]$ lies in the range $\tau \ge \sqrt{2}/\varepsilon$ of (2.19).

For $\tau_m \le \tau \le T$, we have $\tan\varphi = \varepsilon\tau^2/2 \ge 2/\varepsilon \ge 1$, hence

$$\frac{1}{2\tan\varphi} \le \cos\varphi \le \frac{1}{\tan\varphi}, \qquad \frac{1}{4\tan\varphi} \le \frac{\eta_+}{\nu_+} \le \frac{3}{2\tan\varphi},$$

by (2.19). At τ_m, $\tan\varphi \le 9/(2\varepsilon)$, so $\eta_+/\nu_+ \ge \varepsilon/18 > \varepsilon_+$. At $\tau = 2(\varepsilon\varepsilon_+)^{-1/2}$, we have $\tan\varphi = 2/\varepsilon_+$, so $\eta_+/\nu_+ \le 3\varepsilon_+/4 < \varepsilon_+$. Continuity therefore gives a crossing in this interval. At every crossing in $[\tau_m, T]$ the displayed bounds give $1/(4\varepsilon_+) \le \tan\varphi \le 3/(2\varepsilon_+)$, so $\cos\varphi$ is comparable to ε_+.

To estimate the strength at a crossing, we compare the amplitude there with its maximum on $[0,T]$. By (2.18), $|\mathbf{a}|$ lies between $\exp(-\varepsilon)a_{\mathrm{ref}}$ and $\exp(\varepsilon)a_{\mathrm{ref}}$. The maximum of a_{ref} on $[0,T]$ is attained at τ_m, so for $\tau_m \le \tau \le T$, $|\mathbf{a}(\tau)|/\max_{[0,T]}|\mathbf{a}|$ is bounded above and below by absolute multiples of $a_{\mathrm{ref}}(\tau)/a_{\mathrm{ref}}(\tau_m)$. Formula (2.14) gives

$$\frac{a_{\mathrm{ref}}(\tau)}{a_{\mathrm{ref}}(\tau_m)} = \left(\frac{\cos\varphi(\tau)}{\cos\varphi(\tau_m)}\right)^{1/2} \exp\left(\int_{\tau_m}^{\tau} (1+e)\cos\varphi\, d\tau'\right).$$

The integral lies between zero and two, since $\int_{\tau_m}^{\infty} \cos\varphi\, d\tau \le 2/(\varepsilon\tau_m) \le 1$. Since $2/\varepsilon \le \tan\varphi(\tau_m) \le 9/(2\varepsilon)$, $\cos\varphi(\tau_m)$ is comparable to ε. At a crossing $\cos\varphi$ is comparable to ε_+. The preceding amplitude comparison and the normal-length estimate (2.10) now give, at the same time,

$$\frac{|\mathbf{a}(\tau)|}{\max_{[0,T]}|\mathbf{a}|} \asymp \sqrt{\frac{\varepsilon_+}{\varepsilon}}, \qquad |k(\tau)| \asymp \frac{\lambda}{\varepsilon_+}.$$

Here $\asymp$ denotes upper and lower bounds by absolute constant multiples. Multiplying the two estimates gives (2.23).

For $0 \le \tau \le 1$, the bound $Y \le 1$ gives $a_{\mathrm{ref}}(\tau) \le \exp(\tau) \le \exp(1)$, while $\log a_{\mathrm{ref}}(\tau_m) \ge \varepsilon^{-1/2} - 1$ by Lemma 2.5. By (2.18) at such τ and at τ_m, $|\mathbf{a}(\tau)| \le \exp(1+\varepsilon)$ and $|\mathbf{a}(\tau_m)| \ge \exp(\varepsilon^{-1/2} - 1 - \varepsilon)$. Since $2\varepsilon \le 1$, this gives (2.24). □

3. Localized waves over a smooth parent

The affine solution has one normal and one amplitude at each time. In a nonuniform periodic flow, two particles encounter different velocity gradients. We therefore transport a phase and solve an amplitude equation at each particle label. Cutting the resulting wave off in the labels confines it to a material neighborhood of the origin. The leading transport cancellation survives on each trajectory. Spatial derivatives across neighboring trajectories still enter both the divergence and the momentum residual.

3.1. **Lagrangian coordinates and the leading wave.** Let $I = [0,T]$, $T > 0$, and let U be a smooth odd divergence-free velocity on $I \times \mathbb{T}^3$. We need only its flow X and its velocity gradient. Write a for a particle's position at time zero and $x = X(t,a)$ for its position at time t. Set

$$X(0,a) = a, \qquad M = (D_x U) \circ X, \qquad \mathbf{d} = (D_a X)^{-T} \nabla_a.$$

Time derivatives in these coordinates hold a fixed. On the lift to $\mathbb{R}^3$, $X(t, a+n) = X(t,a) + n$ for $n \in \mathbb{Z}^3$.

Fix the time-zero wave vector $k_* \in \mathbb{R}^3 \setminus \{0\}$ and give the particle with label a the phase $\phi(a) = k_* \cdot a$. The transported phase $\phi \circ X(t,\cdot)^{-1}$ has Eulerian gradient $\kappa = \mathbf{d}\phi = (D_a X)^{-T} k_*$ at $X(t,a)$. For a flow linear in a the transported phase is $\kappa \cdot x$, as in Lemma 2.1. A profile $f(t,a,s)$ has two spatial arguments: the initial particle position a and an independent periodic phase $s \in \mathbb{R}/\mathbb{Z}$, measured in cycles. Its value in the fluid is obtained by setting $s = \phi(a)$ and $x = X(t,a)$. Denote the resulting field by $f^\natural$:

$$f^\natural(t, X(t,a)) = f(t,a,\phi(a)), \qquad \widetilde{\mathbf{d}} = \mathbf{d} + \kappa \partial_s.$$

For example, the profile called $A_1(t,x,s)$ in the introduction is written here as $A_1(t, X(t,a), s)$ when pulled back to the labels. This substitution still leaves s free. Evaluation at $s = k_* \cdot a$ is a separate operation.

Before evaluation $\mathbf{d}$ is the Eulerian spatial derivative at fixed phase, written in labels. After evaluation a change of position also changes the phase by its gradient κ, which gives the additional derivative $\kappa \partial_s$. Time differentiation along X keeps both a and its phase fixed. Thus

$$\nabla_x f^\natural = (\widetilde{\mathbf{d}} f)^\natural, \qquad (\partial_t + U \cdot \nabla_x) f^\natural = (f_t)^\natural. \tag{3.1}$$

The phase modulo one is periodic in a exactly when $k_* \in \mathbb{Z}^3$. To allow arbitrary k_*, we evaluate the oscillatory profiles on the lifted label cell $[-1/2, 1/2]^3$ and extend periodically. They vanish near the cell boundary, so the extension is smooth. Phase-independent profiles are already periodic in a.

Along each particle path the local velocity gradient M plays the role of the matrix D in (2.1). Indeed, differentiating $\kappa = (D_a X)^{-T} k_*$ and using $\partial_t D_a X = M D_a X$ gives $\kappa_t = -M^T \kappa$. Choose a smooth envelope $g(a)$ supported in the interior of the label cell and an *activation time* $t_0 \in [0,T)$. On a neighborhood of $\operatorname{supp} g$ the amplitude, called the *seed*, solves the other Craik–Criminale equation,

$$\mathbf{a}_t = -M\mathbf{a} + 2\kappa \frac{\kappa \cdot M\mathbf{a}}{|\kappa|^2}, \qquad \mathbf{a}(t_0,a) = \mathbf{a}_*(a), \qquad \kappa(t_0,a) \cdot \mathbf{a}_*(a) = 0. \tag{3.2}$$

At activation $\mathbf{a}_*(a)$ is the amplitude at $X(t_0, a)$ of the particle that started at a. Different labels may receive different amplitudes. The first wave has constant $\mathbf{a}_*$ and $t_0 = 0$. By (2.1), the seed remains perpendicular to κ for $t_0 \le t \le T$.

Let F_0 be a smooth odd 1-periodic profile and P_0 its zero-mean periodic primitive. The *leading oscillation* and its pressure are

$$A_1 = g\mathbf{a}F_0, \qquad \mathsf{p}[A_1] = -p_{\mathrm{w}} g P_0, \qquad p_{\mathrm{w}} = 2\frac{\kappa \cdot M\mathbf{a}}{|\kappa|^2}. \tag{3.3}$$

Along the particle with label a the leading oscillation is $A_1^\natural(t, X(t,a)) = g(a)\mathbf{a}(t,a)F_0(\phi(a))$. Its value on this particle involves only the amplitude at a, but its spatial derivative also uses the neighboring labels. Componentwise the first chain rule in (3.1) gives

$$\partial_{x_j} A_{1,i}^\natural = \left[\mathbf{d}_j (g\mathbf{a})_i \, F_0 + (g\mathbf{a})_i \, \kappa_j F_0'\right]^\natural.$$

Here the first term differentiates the envelope and the amplitude and the second differentiates the periodic profile. Multiplying (3.2) by gF_0 gives $A_{1,t} + MA_1 = p_{\mathrm{w}} \kappa g F_0$, and $P_0' = F_0$ gives

$$A_{1,t} + MA_1 + \kappa \partial_s \mathsf{p}[A_1] = 0. \tag{3.4}$$

Before the divergence correction or the time cutoff, the momentum equation has just two additional error terms. Since $A_1 \cdot \kappa = 0$, the fast derivative in self-advection drops out. Equation (3.4) cancels the acceleration along the parent, the interaction MA_1, and the fast pressure gradient. The momentum defect of $U + A_1^\natural$, in addition to that of U, is therefore

$$\left((A_1 \cdot \mathbf{d})A_1 + \mathbf{d}\,\mathsf{p}[A_1]\right)^\natural.$$

The same transversality cancels only the fast part of the divergence, leaving $(\mathbf{d}\cdot A_1)^\natural$. We first repair this divergence locally. For any transverse profile V with zero phase average, seek a potential Q whose Eulerian curl has V as its fast part. The curl is $(\widetilde{\mathbf{d}} \times Q)^\natural$ by (3.1). To achieve $\kappa \times \partial_s Q = V$, take

$$Q[V] = -\partial_s^{-1} \frac{\kappa \times V}{|\kappa|^2}, \qquad C[V] = \mathbf{d} \times Q[V], \tag{3.5}$$

where ∂_s^{-1} is the zero-mean periodic primitive in s. Since $\kappa \cdot V = 0$, the vector triple product gives $\kappa \times \partial_s Q[V] = V$. The other part of the curl is $C[V]$, the *divergence correction*. Its component along κ determines how it differentiates the phase of another oscillation. The identities for the completed velocity and for this component are

$$V + C[V] = \widetilde{\mathbf{d}} \times Q[V], \qquad \kappa \cdot C[V] = -\partial_s^{-1}(\mathbf{d} \cdot V). \tag{3.6}$$

For the second identity, apply $\widetilde{\mathbf{d}}\cdot$ to the first. The $\mathbf{d}_i$ commute as pulled-back coordinate derivatives. Since $\kappa_j = \mathbf{d}_j \phi$ and κ is independent of s, the $\widetilde{\mathbf{d}}_i$ commute as well. Thus both $\widetilde{\mathbf{d}} \cdot (\widetilde{\mathbf{d}} \times Q[V])$ and $\mathbf{d} \cdot C[V]$ vanish. Transversality leaves $\partial_s(\kappa \cdot C[V]) = -\mathbf{d} \cdot V$. Taking the zero-mean primitive gives (3.6), since $Q[V]$ and $C[V]$ have zero phase mean.

For the leading oscillation the potential is

$$Q[A_1] = gP_0 \frac{\mathbf{a} \times \kappa}{|\kappa|^2}.$$

The formula for $Q[A_1]$ contains one inverse power of $|\kappa|$. Its phase primitive acts on a period of length one. Taking the curl at fixed phase then differentiates the label dependence of the envelope, amplitude and normal. The resulting one-derivative estimate is (5.31). Both $Q[V]$ and $C[V]$ vanish outside the label support of V at each time, since the primitive acts at fixed label and the curl is local in a. Thus the corrected velocity $(A_1 + C[A_1])^\natural$ is a curl supported in $X(t, \operatorname{supp} g)$. The leading pressure $\mathsf{p}[A_1]^\natural$ also vanishes outside this set. In the construction g is even and equals one

near zero, and $\mathbf{a}_*$ is even in a. The odd flow gives even M and κ, so the seed is even in a. Since F_0 is odd and P_0 is even, the completed Eulerian velocity is odd.

A later wave must switch on at its activation time. Take a smooth time cutoff $0 \le \widetilde{\chi} \le 1$ on $[0,T]$ vanishing for $t < t_0$, with $\widetilde{\chi} \equiv 1$ for the first wave. Set

$$V_1 = \widetilde{\chi} A_1, \qquad \mathsf{p}_1 = \widetilde{\chi}\mathsf{p}[A_1], \qquad C_2 = \widetilde{\chi} C[A_1] = C[V_1]. \tag{3.7}$$

When $t_0 > 0$ the cutoff and all of its derivatives vanish at t_0. The products therefore extend smoothly by zero to the earlier times, where the seed need not be defined. Multiplication by a time cutoff preserves the spatial divergence cancellation. Its derivative along the parent flow contributes the term

$$R_{\widetilde{\chi}} = \widetilde{\chi}' \left(A_1 + C[A_1]\right) \tag{3.8}$$

to the residual, so the cutoff error depends on the size of the wave while it is being switched on.

3.2. **The profile.** At the fixed central particle the evaluated wave has zero velocity, since $F_h(0) = 0$. In its gradient this factor removes the term involving derivatives of the envelope and amplitude. Taking $g(0) = 1$ and $F_h'(0) = 1$ gives

$$\nabla_x A_1^\natural(t,0) = \mathbf{a}(t,0) \otimes \kappa(t,0).$$

This is the leading central gradient. Contributions from the divergence and other corrections are estimated in (5.50). On the rest of the support we need two more bounds on the same profile. Small $\|F_h\|_\infty$ keeps the velocity small. The lower bound $F_h' \ge -Ch$ controls positive pressure curvature, since the leading normal curvature has the sign of $-p_{\mathrm{w}} F_h'$ where $p_{\mathrm{w}} > 0$. We use the following F_h for F_0.

Lemma 3.1. *There are odd smooth 1-periodic functions F_h, $0 < h \le 1$, with zero-mean periodic primitives P_h, such that*

$$F_h'(0) = 1, \qquad -Ch \le F_h' \le 1, \qquad \|F_h\|_\infty \le Ch, \qquad \left\|\partial_s^j F_h\right\|_\infty \le C_j h^{1-j} \quad (j \ge 2).$$

Here j is an integer, and the constants $C > 0$ and $C_j > 0$ are independent of h.

Proof. Fix an even $\psi \in C_c^\infty((-1/4, 1/4))$ with $0 \le \psi \le 1$ and $\psi(0) = 1$. The bump $\psi(s/h)$ has integral $c_h = h \int_{\mathbb{R}} \psi < h/2$ over one period. Subtract its mean to obtain a periodic primitive, and normalize the slope at zero by setting

$$F_h'(s) = \frac{\psi(s/h) - c_h}{1 - c_h} \quad (|s| \le 1/2), \qquad F_h(0) = 0.$$

The derivative is even with mean zero, so F_h is odd and periodic and has a zero-mean periodic primitive P_h. Since $c_h < 1/2$, the slope equals one at zero and lies between $-2c_h$ and one. Integrating the lower bound from 0 to s and from s to 1, using $F_h(0) = F_h(1) = 0$, gives $\|F_h\|_\infty \le 2c_h$. Differentiating $\psi(s/h)$ another $j-1$ times gives the bound $C_j h^{1-j}$. □

4. Selecting initial data by the endpoint problem

Given a smooth force f and a parent velocity u, we seek a smooth divergence-free correction b satisfying the linearized momentum equation (1.6) with some pressure q. The initial velocity $b(0)$ is unknown. We determine it by requiring that every particle have zero first-order displacement at each end of the interval. We need this operation for the phase average of a wave error even when the parent itself has a residual. Theorem 4.2 constructs the correction and bounds the inverse in spatial Hölder norms by a polynomial in the parent coefficients.

Fix $0 < T \le 1$, $I = [0,T]$, and a smooth divergence-free periodic parent u. Let $X(t,a)$, $X(0,a) = a$, be its volume-preserving flow. The gradient M of u along a trajectory and the coefficient $H^\#$ of its displacement equation depend only on u. In particle labels the metric K measures physical lengths, and Ω contains the skew part of the velocity gradient. The four matrices are

(4.1) $M = (\nabla u) \circ X, \quad H^{\#} = -(M_t + M^2), \quad K = (D_a X)^T D_a X, \quad \Omega = (D_a X)^T (M^T - M) D_a X.$

Time derivatives keep a fixed. Choose $D_- \geq \max\{1, \sup_{I,a} |(D_a X)^{-1}|\}$. The flow equation gives

$$\partial_t D_a X = M D_a X, \qquad K_t - \Omega = 2(D_a X)^T \partial_t D_a X. \tag{4.2}$$

Let p be a smooth function and write $R = u_t + u \cdot \nabla u + \nabla p$. The gradient of this identity is $(\partial_t + u \cdot \nabla)\nabla u + (\nabla u)^2 + D^2 p = \nabla R$, with $(\nabla u)_{ij} = \partial_j u_i$. Along trajectories the material derivative becomes ∂_t, so

$$H^{\#} = (D^2 p - \nabla R) \circ X. \tag{4.3}$$

Consequently $H^{\#}$ is independent of the choice of p. Adding a function to the pressure adds its gradient to R, and the changes cancel in (4.3). For an exact Euler parent, take its Euler pressure so that $R = 0$. Then $H^{\#}$ is the symmetric pressure Hessian along X. For a general parent its quadratic form still depends only on its symmetric part. The skew matrix in Ω records the vorticity, since $(M^T - M)y = -(\operatorname{curl} u \circ X) \times y$. We shall use the one-sided bound $y \cdot H^{\#} y \leq |y|^2$ to invert the endpoint problem.

4.1. **Displacement and the two end conditions.** Here $H^1_0(I; \mathbb{R}^3)$ is the space of $z \in H^1(I; \mathbb{R}^3)$ with $z(0) = z(T) = 0$, the closure of $C^\infty_c((0,T); \mathbb{R}^3)$ in H^1. Fix a label a, let $z \in H^1_0(I; \mathbb{R}^3)$, and put $\xi = D_a X(\cdot, a) z$. Since $\partial_t D_a X = M D_a X$,

$$\xi_t = (\partial_t D_a X) z + D_a X z_t = M\xi + D_a X z_t, \qquad z_t = (D_a X)^{-1}(\xi_t - M\xi). \tag{4.4}$$

For a displacement field induced by a divergence-free correction b, ξ measures the displacement in physical space and $z = (D_a X)^{-1}\xi$ measures it in the parent particle coordinates. Equations (1.5) and (4.4) identify its derivative $v = z_t = (D_a X)^{-1}(b \circ X)$. Volume preservation gives $\operatorname{div}_a z = 0$ for such fields. Since $D_a X$ is invertible, the two physical end conditions become $z(0) = z(T) = 0$, or equivalently $z(0) = 0$ and $\int_0^T v(t,a)\, \mathrm{d}t = 0$ at every label. This vanishing integral in time, separately at every label, is the condition that determines the otherwise free initial velocity. The example with $u = 0$ in Subsection 1.2 shows how its value can depend on forces applied late in the interval.

Cauchy–Schwarz in $z(t) = -\int_t^T z_t(s)\, \mathrm{d}s$ gives

$$|z(t)|^2 \leq (T - t) \int_I |z_t|^2 \, \mathrm{d}s, \qquad \int_I |z|^2 \, \mathrm{d}t \leq \frac{T^2}{2} \int_I |z_t|^2 \, \mathrm{d}t. \tag{4.5}$$

Since $D_a X$ is smooth, $\xi \in H^1_0(I; \mathbb{R}^3)$ as well, so (4.5) also holds for ξ; in particular $\|\xi\|_{L^2(I)} \leq \|\xi_t\|_{L^2(I)}$, since $T \leq 1$. By (4.4) and $|(D_a X)^{-1}| \leq D_-$,

$$\|z_t\|_{L^2(I)} \leq D_- \big(\|\xi_t\|_{L^2(I)} + \|M\|_\infty \|\xi\|_{L^2(I)} \big) \leq D_- (1 + \|M\|_\infty) \|\xi_t\|_{L^2(I)}.$$

Squaring gives

$$\int_I |z_t|^2 \, \mathrm{d}t \leq D_-^2 (1 + \|M\|_\infty)^2 \int_I |\xi_t|^2 \, \mathrm{d}t. \tag{4.6}$$

4.2. **Coercivity.** The label-coordinate equations involve K and Ω, whereas the assumption we can verify is an upper bound on $H^{\#}$. To connect them we rewrite the quadratic form of the label equation in the physical displacement ξ. A boundary term drops out precisely because the displacement vanishes at both ends.

Substitute $D_a X z_t = \xi_t - M\xi$ and $D_a X z = \xi$ in the two terms. First, $K z_t \cdot z_t = |D_a X z_t|^2 = |\xi_t|^2 - 2\xi_t \cdot M\xi + |M\xi|^2$. Second, $\Omega z_t \cdot z = (M^T - M)(\xi_t - M\xi) \cdot \xi = \xi_t \cdot M\xi - \xi_t \cdot M^T \xi - |M\xi|^2 + M^2 \xi \cdot \xi$. Adding, and using $\partial_t(\xi \cdot M\xi) = \xi_t \cdot M\xi + \xi_t \cdot M^T \xi + M_t \xi \cdot \xi$ and $M_t + M^2 = -H^{\#}$, we obtain

$$K z_t \cdot z_t + \Omega z_t \cdot z = |\xi_t|^2 - H^{\#} \xi \cdot \xi - \partial_t(\xi \cdot M\xi).$$

Since ξ vanishes at both ends, integration and density give

$$\int_I (Kz_t \cdot z_t + \Omega z_t \cdot z)\,\mathrm{d}t = \int_I (|\xi_t|^2 - H^{\#}\xi \cdot \xi)\,\mathrm{d}t. \tag{4.7}$$

Lemma 4.1. *Assume $H^{\#} \le \mathrm{Id}$ on $I \times \mathbb{T}^3$ as quadratic forms. Then for every $a \in \mathbb{T}^3$, $z \in H^1_0(I;\mathbb{R}^3)$ and $\xi = D_a X(\cdot, a)z$, one has $\int_I H^{\#}\xi \cdot \xi \,\mathrm{d}t \le \frac{1}{2}\int_I |\xi_t|^2 \,\mathrm{d}t$. Hence*

$$\int_I (Kz_t \cdot z_t + \Omega z_t \cdot z)\,\mathrm{d}t \ge \frac{1}{2}\int_I |\xi_t|^2 \,\mathrm{d}t. \tag{4.8}$$

Proof. The assumption $H^{\#} \le \mathrm{Id}$, the second inequality of (4.5) for ξ and $T \le 1$ give

$$\int_I H^{\#}\xi \cdot \xi \,\mathrm{d}t \le \int_I |\xi|^2 \,\mathrm{d}t \le \frac{T^2}{2}\int_I |\xi_t|^2 \,\mathrm{d}t \le \frac{1}{2}\int_I |\xi_t|^2 \,\mathrm{d}t.$$

Equation (4.7) gives (4.8). □

4.3. **The inverse and its pressure.** Take first a smooth force f and write $g = (D_a X)^T (f \circ X)$ and $\mathsf{q} = q \circ X$, the pressure in particle coordinates. Substituting $b \circ X = D_a X\, v$ into the linearized momentum equation and multiplying by $(D_a X)^T$ gives

$$Kv_t + 2(D_a X)^T (\partial_t D_a X)v + \nabla_a \mathsf{q} = g.$$

By (4.2), the first two terms combine as $\partial_t(Kv) - \Omega v$. Integrating from zero to t and putting $L = -\int_0^t g(s)\,\mathrm{d}s$ gives the system

$$K(t)v(t) - \int_0^t (\Omega v)(s)\,\mathrm{d}s + L(t) = c + \nabla_a \phi, \qquad v = z_t, \quad z(0) = z(T) = 0, \quad \mathrm{div}_a\, z = 0, \tag{4.9}$$

for the displacement z, a time-independent solenoidal field c and a gradient $\nabla_a \phi$. For these force data, $\phi = -\int_0^t \mathsf{q}(s)\,\mathrm{d}s$ up to a spatial constant. Here $L(0) = 0$ and $K(0) = \mathrm{Id}$. Evaluating the equation at time zero and projecting onto divergence-free fields gives $c = v(0) = b(0)$. The integrated equation also makes sense for continuous L with spatial Hölder regularity. Its unknown pressure term is the gradient $\nabla_a \phi$. At $t = 0$ projection now gives $c = v(0) + \mathbb{P}L(0)$. In the smooth-force case differentiation recovers the instantaneous pressure as well.

For smooth forces the desired conclusion is the Eulerian equation (4.12) with both displacement endpoints fixed. We state first the estimate for integrated data. It specifies the coefficient norms and the power that must remain under control when the inverse is used at successive stages.

Theorem 4.2. *Assume $H^{\#} \le \mathrm{Id}$ on $I \times \mathbb{T}^3$ as quadratic forms. For every $L \in C(I; C^{0,1/2})$ the system (4.9) has a unique solution with $z \in C^1(I; C^{0,1/2})$, $c \in C^{0,1/2}$ divergence free and $\nabla_a \phi \in C(I; C^{0,1/2})$. There are absolute C, p such that, with*

$$A = C\big(D_- + \|M\|_\infty + \|K\|_{C_t C^{1,1/2}} + \|\Omega\|_{C_t C^{1,1/2}}\big)^p, \tag{4.10}$$

one has $\|v\|_{C_t C^{0,1/2}} \le A\,\|L\|_{C_t C^{0,1/2}}$.

Let f be a smooth field on $I \times \mathbb{T}^3$, and let $L = -\int_0^t g(s)\,\mathrm{d}s$ with $g = (D_a X)^T (f \circ X)$. Then z is smooth. Put $B = D_a X v$ and $\mathcal{D} = 2(D_a X)^T \partial_t D_a X$. There is a smooth zero-mean label pressure q such that

$$Kv_t + \nabla_a \mathsf{q} = g - \mathcal{D}v. \tag{4.11}$$

Thus $b = B \circ X^{-1}$ and $q = \mathsf{q} \circ X^{-1}$ are smooth and solve

$$\mathrm{div}\, b = 0, \qquad b_t + u \cdot \nabla b + b \cdot \nabla u + \nabla q = f. \tag{4.12}$$

The endpoint conditions select the initial velocity $b(0)$.

The number A also bounds, uniformly in t, the operator norm on $C^{0,1/2}$ of the fixed-time inverse $(\mathbb{P}K(t))^{-1}\mathbb{P}$, where $(\mathbb{P}K(t))^{-1}$ inverts the map $v \mapsto \mathbb{P}(K(t)v)$ on the divergence-free fields of $C^{0,1/2}$.

Proof. Appendix A first inverts an operator on whole paths: there the condition $\int_0^T v(t,a)\,\mathrm{d}t = 0$ at each label and the coercive form can be used together. This controls a time norm. A second, spatial inversion at fixed t gives the velocity estimate uniformly up to the endpoints. Rescale time by $\tau = t/T$ and put $Z(\tau) = z(t)/T$, so $Z_\tau = v$. Theorem A.1 applies with coefficients $K, T\Omega$, ellipticity D_-^{-2} and source L. Its coercivity constant is $1/\big(2D_-^2(1+\|M\|_\infty)^2\big)$, by Lemma 4.1 and (4.6). It gives the unique solution. Since $D_- \geq 1$ and $T \leq 1$, the quantity Q of Appendix A is at most an absolute constant times a fixed power of the sum in parentheses in (4.10), so the estimate (A.3) gives the asserted bound. For a smooth f the source L is smooth, and so is the solution, by the last assertion of Theorem A.1. Differentiating (4.9), using $K_t - \Omega = \mathcal{D}$ from (4.2), and setting $\mathsf{q} = -\phi_t$, where ϕ has zero spatial mean by the pressure convention of Section 1, gives (4.11). Then $B_t + MB = D_a X v_t + 2(\partial_t D_a X)v$ and the Piola identity $\mathrm{div}_x(B \circ X^{-1}) = (\mathrm{div}_a\, v) \circ X^{-1}$ give (4.12). Finally, by Lemma A.2 with $H = \mathbb{R}^3$, $\mathcal{R} = \mathbb{P}$ and $B = K(t)$, whose coercivity is D_-^{-2}, the fixed-time inverse is bounded by a fixed power of $2 + D_-^2 + \|K(t)\|_{C^1}$. Since $D_- \geq 1$, this number is also at most an absolute constant times a fixed power of the sum in parentheses in (4.10), so A bounds the fixed-time inverse once C and p are enlarged. □

5. From a localized wave to an approximate solution

Fix a parent velocity U of finite Fourier support. Its Euler residual R_U, its flow deformation and its gradient enter every estimate for the correction. At the origin the selected wave must dominate the growth of all transverse solutions up to a parent factor, and its first coupling must stay positive (5.5). The mean equation requires $D^2 p_U - \nabla R_U \leq \mathrm{Id}$. For a nonzero inherited residual the admissible frequencies satisfy

$$(\Gamma_P \mathsf{D}^b)^{12} \leq \lambda \leq \|R_U\|_{C_t L_x^\infty}^{-2/5}.$$

The left bound pays for the parent's derivative and deformation bounds and for the depth P of the expansion. The right bound makes the parent residual small at the squared velocity scale of the new wave. For the stationary starting shear $R_U = 0$, only the lower bound applies. Equation (7.8) and the preceding residual estimate in (T2) give room at both ends for every subsequent wave.

Three estimates determine whether the corrected field can serve as the next parent. First, the mean, transverse oscillation and curl correction at each level reduce the momentum residual. Theorem 5.6 estimates the finite expansion. In physical space the depth and activation errors have sizes $\lambda^{-2P/3}$ and $\|\widetilde{\chi}'\gamma\|_\infty$, with common factor $C_{\mathrm{step}}\lambda^2$ in (5.48). Second, the central gradient must retain the leading shear while the positive pressure curvature stays small across the torus. Lemma 5.9 gives both bounds from the profile slopes. Third, the derivatives of the flow map and its inverse enter the next endpoint problem. Estimating label and phase independently gives a bound $C_{\mathrm{step}}\lambda^{1/4}$ for the new deformation (Lemma 5.10). Truncating the corrected field in frequency then gives a new parent. Proposition 5.11 controls the changes in residual and pressure made by that truncation.

5.1. **A step over a truncated parent.**

Here $[0,T]$ is a physical time interval. Every phase mean is solved on all of it, even if the oscillation starts at $t_0 > 0$. For $j \geq 2$ we fix $T = S_j$, choose the wave normal and growing direction at the central particle, and normalize the amplitude on $[t_j, S_j]$. This rescaling leaves the coupling ratio unchanged, so we can then use its crossing to choose the next launch.

We use the Fourier modes $e^{2\pi i \xi\cdot x}$ for $\xi \in \mathbb{Z}^3$. A field is *supported on frequencies at most* Λ if its Fourier coefficients vanish for $|\xi| > \Lambda$. Let the parent U be a smooth odd divergence-free field on $[0,T]\times\mathbb{T}^3$, $T \leq 1$, and use its flow X, the matrix M and the operators $\mathbf{d}$, $\widetilde{\mathbf{d}}$ of Subsection 3.1. Let $\mathsf{D} \geq 2$ be a number such that U is supported on frequencies at most $2\mathsf{D}$ and

$$\|\nabla U\|_\infty \leq \mathsf{D}, \qquad \sup|(D_a X)^{\pm 1}| \leq \mathsf{D}. \tag{5.1}$$

Since U is odd, the flow fixes the origin, and D_aX and M are even in a. Let p_U be the function of zero mean with $\Delta p_U = -\operatorname{tr}((\nabla U)^2)$, and let $R_U = U_t + U\cdot\nabla U + \nabla p_U$ be the residual of the parent; no other equation is assumed. Thus $R_U = U_t + \mathbb{P}\operatorname{div}(U\otimes U)$ is odd, divergence free and supported on frequencies at most $4\mathsf{D}$. To apply the endpoint inverse of Theorem 4.2 to the mean correction over U, assume that

$$D^2 p_U - \nabla R_U \le \mathrm{Id} \qquad \text{on } [0,T]\times\mathbb{T}^3 \tag{5.2}$$

as quadratic forms. By (4.3) this is the hypothesis $H^{\#}\le \mathrm{Id}$ of that theorem.

For the wave of Subsection 3.1, take a time cutoff $\widetilde{\chi}$ with activation time $0\le t_0<T$ and a constant $k_*\neq 0$ such that $|\kappa(t_0,0)|=\lambda\ge 1$. Since $\kappa = (D_aX)^{-T}k_*$, (5.1) gives

$$\mathsf{D}^{-1}\lambda \le |k_*| \le \mathsf{D}\lambda, \qquad \mathsf{D}^{-2}\lambda \le |\kappa| \le \mathsf{D}^2\lambda. \tag{5.3}$$

Take $F_0 = F_h$ from Lemma 3.1 with $h=\lambda^{-1/2}$. At $a=0$, solve (3.2) on $[t_0,T]$ with $\mathbf{a}\perp\kappa$ and normalize $\max|\mathbf{a}(t,0)| = \lambda^{-3/4}$. A nonzero solution of the linear seed equation never vanishes, so we can put

$$\gamma(t) = \lambda^{3/4}|\mathbf{a}(t,0)|, \qquad \nu(t,0) = \Big(\frac{\kappa}{|\kappa|}\cdot M\frac{\mathbf{a}}{|\mathbf{a}|}\Big)(t,0) \qquad (t_0\le t\le T).$$

The first coupling is $\nu(t,0)$ and the strength is $|\mathbf{a}(t,0)||\kappa(t,0)|$, as in Lemma 2.2. At other labels solve the same equation with initial data

$$\mathbf{a}(t_0,a) = \big(\mathrm{Id} - P_{\kappa(t_0,a)}\big)\mathbf{a}(t_0,0), \qquad P_\kappa = \frac{\kappa\otimes\kappa}{|\kappa|^2}, \tag{5.4}$$

the projection of the vector at the origin onto the plane perpendicular to $\kappa(t_0,a)$. It is even in a.

Choose a smooth $\chi:[0,\infty)\to[0,1]$ equal to one on $[0,1]$ and zero on $[2,\infty)$. The even envelope $g(a)=\chi(|a|/\ell_*)$ keeps the seed intact for $|a|\le\ell_*$ and cuts it off for $|a|\ge 2\ell_*$. Thus $A_1 = g\mathbf{a}F_h$ in (3.3) and $V_1 = \widetilde{\chi}A_1$ in (3.7).

At the origin, assume that for $t_0\le t'\le t\le T$,

$$\frac{|\mathbf{b}(t)|}{\gamma(t)} \le \mathsf{D}\,\frac{|\mathbf{b}(t')|}{\gamma(t')}, \qquad \nu(t,0)\ge \mathsf{D}^{-1}, \tag{5.5}$$

the first inequality holding for every transverse solution $\mathbf{b}$ at the origin. On a small enough envelope, comparison with the central solution extends both the relative growth bound and the strict sign $\kappa\cdot M\mathbf{a}>0$ to every label there (Lemma 5.3). Rescaling k_* by a positive factor preserves the normal direction and the transverse amplitude equation. Normalizing the seed changes neither its direction nor the relative growth $\gamma(t)/\gamma(t')$. These hypotheses can therefore be checked before choosing the frequency and depth.

The radius ℓ_* measures the envelope in *labels*. Within it, a unit of phase has local normal length $|\kappa|^{-1}$ in physical space, and the steep part of the profile has thickness $h/|\kappa|$. At launch at the origin these lengths are λ^{-1} and $\lambda^{-3/2}$. The much finer Fourier cutoff at λ^5 will be applied to the completed velocity in Section 7. By (5.19) the seed on the envelope is at most twice its size at the origin. Multiplication by F_h, with $\|F_h\|_\infty\le Ch$, bounds the velocity by $C\lambda^{-5/4}$. The first mean must cancel the averaged quadratic forcing and R_U. Averaging turns the forcing in (5.25) into a derivative of the stress at fixed phase. This leaves the square of the velocity scale, up to parent factors, so we require

$$\|R_U\|_{C_tL^\infty} \le \lambda^{-5/2}. \tag{5.6}$$

Increasing λ makes this bound on the inherited residual more stringent. At wave j, item (T2) of Theorem 7.1 at step $j-1$ already gives $\|R_{j-1}\|_{C_tL^\infty}\le\lambda_j^{-3}$. The radius of the envelope is a negative power of D with an absolute exponent, $\ell_*\le\mathsf{D}^{-3}$. The exponent is fixed sufficiently large for the finitely many estimates of Subsections 5.2–5.4. In particular the support of g lies in the interior of $[-1/2,1/2]^3$.

The constants and exponents C are absolute. Constants C_n may depend on the order, the fixed profiles and the corresponding constants in the hypotheses. The indices p and P allow dependence on level and depth as well. No growth bound in these indices is needed. Equation (7.8) shows how the iteration absorbs the fixed-order constants and the powers of D.

5.2. **Coefficient bounds for the parent.** The coefficients of the mean and transverse equations include D_aX, $M = (\nabla U)\circ X$ and κ. Their ordinary label derivatives grow with the order. To obtain a fixed exponent of the parent bound D, measure a profile in units of the envelope radius ℓ_* and the phase width h, as in (D.1),

$$\|f\|_{[n]} = \sum_{|\beta|+r\le n} \ell_*^{|\beta|} h^r \sup|\partial_a^\beta \partial_s^r f|, \qquad \|f\|_{[n],\gamma} = \|f/\gamma\|_{[n]}.$$

The supremum includes time. We use $[0,T]$ for the unweighted estimates unless a smaller interval is indicated, and $[t_0,T]$ for the weighted ones. Only label and phase derivatives enter. For a coefficient $c(t,a)$ the norm reduces to $\sum_{m\le n}\ell_*^m|c|_m$, where $|c|_m$ sums the suprema of its label derivatives of order m. A single choice of ℓ_* will absorb a factor D^C for each label derivative.

Proposition 5.1. *There is an absolute exponent C such that each coefficient c among*

$$(D_aX)^{\pm1}, \qquad M, \qquad \kappa/\lambda, \qquad \lambda^2/|\kappa|^2, \qquad P_\kappa$$

satisfies (D.6). Let $f(t,x)$ be a field, smooth in x and with values in a normed space, such that $|f|_m \le C'_m A_f \mathsf{D}^m$ for every $m\ge 0$ and a number $A_f\ge 0$. Then

$$|f\circ X|_m \le C_m A_f \mathsf{D}^{Cm} \qquad (m\ge 0). \tag{5.7}$$

Consequently, for every $n\ge 0$,

$$\|c\|_{[n]} \le C_n \mathsf{D}^C, \qquad \|f\circ X\|_{[n]} \le C_n A_f. \tag{5.8}$$

Proof. When differentiating the flow equation, the only term containing the unknown derivative of highest order is linear in that derivative. Its propagator is the known first derivative of the flow between the two times, of norm at most D^2 by (5.1). For $n\ge 1$ let $\upsilon_n = \max_i |X_i|_n$, so $\upsilon_1\le 3\mathsf{D}$. By (B.1) for ∇U, with values in the continuous functions of time, and (5.1), $|U|_k \le C_k\mathsf{D}^{k-1}\|\nabla U\|_\infty \le C_k\mathsf{D}^k$ for $k\ge 1$. For $|\beta|=n\ge 2$, the chain rule gives

$$\partial_t\partial^\beta X = (\nabla U\circ X)\partial^\beta X + N_\beta,$$

where N_β is a sum, with coefficients that depend only on β, of the terms

$$\big((\partial^{\beta'}U)\circ X\big)\prod_{j=1}^{k}\partial^{\beta_j}X_{l_j}, \qquad 2\le k=|\beta'|\le n, \quad \beta_1+\cdots+\beta_k=\beta, \quad |\beta_j|\ge 1.$$

Since $\partial_t D_aX = (\nabla U\circ X)D_aX$ and $\partial^\beta X(0)=0$,

$$\partial^\beta X(t) = \int_0^t D_aX(t)(D_aX(\sigma))^{-1}N_\beta(\sigma)\,\mathrm{d}\sigma.$$

By (5.1) the propagator has norm at most D^2, and $T\le 1$. Hence, for $n\ge 2$,

$$\upsilon_n \le C_n\mathsf{D}^2\sum_{k=2}^{n}\mathsf{D}^k\sum_{n_1+\cdots+n_k=n}\upsilon_{n_1}\cdots\upsilon_{n_k}, \tag{5.9}$$

with every $n_i\ge 1$. Induction on n gives $\upsilon_n\le C_n\mathsf{D}^{5n-4}$: by the bounds at the orders $n_i<n$, a term of (5.9) with k factors is at most C_n times

$$\mathsf{D}^{2+k}\,\mathsf{D}^{5n-4k} = \mathsf{D}^{5n-4-3(k-2)} \le \mathsf{D}^{5n-4}.$$

Thus $|D_aX|_m \le C_m\mathsf{D}^{5m+1}$ for every $m \ge 0$. For f as in the statement, the chain rule bounds $|f\circ X|_n$, $n \ge 1$, by C_n times the terms $A_f\mathsf{D}^k\mathsf{D}^{5n-4k}$ with $1 \le k \le n$, which proves (5.7). By (B.1), $f = \nabla U$ satisfies the hypothesis with $A_f = \|\nabla U\|_\infty \le \mathsf{D}$, which bounds M.

The product and inverse rules of Lemma D.2(a),(b) now apply successively to $(D_aX)^{-1}$, $\kappa/\lambda = (D_aX)^{-T}k_*/\lambda$, $|\kappa/\lambda|^{-2}$ and P_κ. Indeed, (5.1) bounds the inverse deformation, and (5.3) gives $|k_*|/\lambda \le \mathsf{D}$ and $|\kappa|/\lambda \ge \mathsf{D}^{-2}$. Finally, ℓ_* is at most the power D^{-C} that occurs here. So part (c) gives the first bound of (5.8), and $\ell_*^m|f\circ X|_m \le C_mA_f(\mathsf{D}^C\ell_*)^m \le C_mA_f$ gives the second. □

The parent residual has frequencies at most $4\mathsf{D}$. Bernstein's inequality (B.1) therefore gives the hypothesis of (5.7) with $A_f = \|R_U\|_{C_tL^\infty}$. By (5.6) the force that the first mean must cancel satisfies, for every $n \ge 0$,

$$\|R_U\circ X\|_{[n]} \le C_n\lambda^{-5/2}. \tag{5.10}$$

5.3. **Finite-order estimates for the mean.** The phase-independent corrections of the expansion solve the linearized Euler equations over the parent, with zero displacement at both ends of $[0,T]$. For a smooth phase-independent source $\bar f(t,a)$, periodic in a, Theorem 4.2 over U gives $(B,\bar q)$ with

$$B_t + MB + \mathbf{d}\bar q = \bar f, \qquad \mathbf{d}\cdot B = 0, \qquad z(0) = z(T) = 0, \quad B = D_aXz_t. \tag{5.11}$$

Here $\bar f = f\circ X$, $v = z_t$ and $\bar q$ is the pressure in (4.11). The datum for the integrated equation (4.9) is $L(t,a) = -\int_0^t D_aX(s,a)^T\bar f(s,a)\,\mathrm{d}s$. It selects $B(0)$ so that $\int_0^T(D_aX)^{-1}B\,\mathrm{d}t = 0$ at each label.

The example with zero background in Subsection 1.2 shows why a late source can select a nonzero $B(0)$. The inverse preserves order in the scaled Hölder norms on $[0,T]$. The extra source derivative below bounds the Hölder seminorm by supremum norms in (D.2).

Proposition 5.2. *There is an absolute exponent C such that, for every $n \ge 0$, every solution of the mean equation (5.11) satisfies*

$$\|B\|_{[n]} \le C_n\mathsf{D}^C\left\|\bar f\right\|_{[n+1]}. \tag{5.12}$$

Proof. By Proposition 5.1 and Lemma D.2(a), the coefficients K and Ω of (4.1), which enter (4.9), satisfy (D.6). Theorem 4.2 applies over U with $D_- = \mathsf{D}$, by (5.1). Its constant A of (4.10) contains D_-, $\|M\|_\infty \le \mathsf{D}$ and the norms of K and Ω in $C_tC^{1,1/2}$, so it is at most D^C. Since ℓ_* is a sufficiently large negative power of D, the first bound of (D.2) and Lemma D.2(c) give

$$\ell_*^m\big(|K|_{m,1/2} + |\Omega|_{m,1/2}\big) \le C_m\mathsf{D}^C\ell_*^{1/2} \le C_m/A \qquad (m \ge 1). \tag{5.13}$$

Label derivatives preserve the endpoint and divergence constraints, the time independence of the divergence-free field, and the class of label gradients in (4.9). Theorem 4.2 bounds the map $L \mapsto v$ by A in $C_tC^{0,1/2}$. The term Kv enters directly, and Ωv enters through time integration, an operator of norm at most $T \le 1$. Thus (5.13) gives the coefficient bounds for Lemma D.3 with $C_{\mathrm{low}} = A$. Applying its Hölder estimate to the source $L = -\int_0^t g(t')\,\mathrm{d}t'$ of (4.9) gives

$$\|v\|_{[n],1/2} \le C_n\mathsf{D}^C\|g\|_{[n],1/2}. \tag{5.14}$$

By (D.2) and (5.8), $\|D_aX\|_{[n],1/2} \le C_n\ell_*^{-1/2}\mathsf{D}^C \le C_n\mathsf{D}^{C'}$, with C' absolute. Using $B = D_aXv$ and $g = (D_aX)^T\bar f$ in (5.14), the product estimate (D.4) gives $\|B\|_{[n],1/2} \le C_n\mathsf{D}^C\|\bar f\|_{[n],1/2}$. The comparison (D.2) now gives (5.12). □

5.4. **The transverse equation.** Along the parent trajectory $x = X(t,a)$ the physical phase stays equal to $k_*\cdot a$. We solve for the profile at each fixed pair (a,s) and evaluate it at $s = k_*\cdot a$ afterward. The oscillatory correction starts from zero, hence vanishes until its forcing starts. On $[t_0,T]$ we measure it relative to the growing seed by dividing by $\gamma(t)$.

We call

$$V_t + MV + \kappa\partial_s\mathsf{p} = f, \qquad \kappa\cdot V = 0, \qquad \langle V\rangle_s = \langle f\rangle_s = 0, \tag{5.15}$$

the *transverse equation*, because its unknown V is perpendicular to the wave vector κ. The pressure term keeps V perpendicular to κ: differentiating $\kappa \cdot V = 0$ and using $\kappa_t = -M^T\kappa$ in (5.15) gives

$$|\kappa|^2 \partial_s \mathsf{p} = \kappa \cdot f - 2\kappa \cdot MV. \tag{5.16}$$

The transverse equation is therefore the linear ordinary differential equation

$$V_t = -(\mathrm{Id} - 2P_\kappa)MV + (\mathrm{Id} - P_\kappa)f \tag{5.17}$$

at each fixed (a, s); for $f = 0$ it is the seed equation (3.2). For every source we take its solution with $V(0, a, s) = 0$, and p is the zero-mean primitive in s of (5.16). For smooth f the solution is unique and smooth, and V satisfies the two constraints in (5.15). Both V and p vanish at every label where f vanishes for all (t, s). If f vanishes for $t < t_0$, then $V = 0$ for $t \le t_0$.

The plane $\kappa^\perp$ varies with the label. To compare the equation on the envelope with the one at the origin and to differentiate it in the labels, we express V in coordinates in a fixed $\mathbb{R}^2$. Choose a constant 3×2 matrix with orthonormal columns spanning $k_*^\perp$, let $A_\perp$ be D_aX times this matrix, and let $A_\perp^\dagger = (A_\perp^T A_\perp)^{-1} A_\perp^T$. The columns of $A_\perp$ span $\kappa^\perp$, since $\kappa = (D_aX)^{-T} k_*$, and $A_\perp^\dagger$ takes a vector to its coordinates in those columns after orthogonal projection onto their span. Thus $A_\perp^\dagger A_\perp = \mathrm{Id}_{\mathbb{R}^2}$ and $A_\perp A_\perp^\dagger = \mathrm{Id} - P_\kappa$. Here v in $V = A_\perp v$ denotes the two coordinates of the transverse velocity. Since $\partial_t A_\perp = MA_\perp$, the equation (5.17) becomes

$$v_t + \mathcal{M}_\perp v = \psi, \qquad \mathcal{M}_\perp = 2A_\perp^\dagger M A_\perp, \qquad \psi = A_\perp^\dagger f. \tag{5.18}$$

Lemma 5.3. *For $t_0 \le t \le T$ and $|a| \le 2\ell_*$, the seed satisfies*

$$|\mathbf{a}(t, a)| \le 2|\mathbf{a}(t, 0)|, \qquad (\kappa \cdot M\mathbf{a})(t, a) > 0. \tag{5.19}$$

There is an absolute exponent C such that the following holds at every order $n \ge 0$.

(i) Let V and f be smooth fields on $[t_0, T]$ that satisfy (5.17), with V perpendicular to κ, and suppose that $V(t_0)$ and f vanish at the labels outside $\operatorname{supp} g$. *Then*

$$\|V\|_{[n],\gamma} \le C_n \mathsf{D}^C \big(\gamma(t_0)^{-1} \|V(t_0)\|_{[n]} + \|f\|_{[n],\gamma}\big). \tag{5.20}$$

(ii) Let f be smooth on $[0, T]$ with zero phase mean, and suppose that f vanishes for $t < t_0$ and at the labels outside $\operatorname{supp} g$. *The solution of (5.15) with $V(0) = 0$ and its pressure satisfy*

$$\|V\|_{[n],\gamma} + \lambda \|\mathsf{p}\|_{[n],\gamma} \le C_n \mathsf{D}^C \|f\|_{[n],\gamma}. \tag{5.21}$$

(iii) On $[t_0, T]$ the leading oscillation and its pressure satisfy

$$\|A_1\|_{[n],\gamma} + \lambda \|\mathsf{p}[A_1]\|_{[n],\gamma} \le C_n \mathsf{D}^C \lambda^{-5/4}. \tag{5.22}$$

Proof. By (5.1), $\mathsf{D}^{-1}|w| \le |A_\perp w| \le \mathsf{D}|w|$ for $w \in \mathbb{R}^2$. Hence $|A_\perp| \le \mathsf{D}$ and $|A_\perp^\dagger| \le \mathsf{D}$, and the matrix $K_\perp = A_\perp^T A_\perp$ satisfies $\sup |K_\perp^{-1}| \le \mathsf{D}^2$. By Proposition 5.1 and Lemma D.2(a),(b), the coefficients $A_\perp$, $A_\perp^\dagger = K_\perp^{-1} A_\perp^T$ and $\mathcal{M}_\perp$ satisfy (D.6).

In the coordinates $V = A_\perp v$, the seed equation at the label a is $v_t + \mathcal{M}_\perp(t, a)v = 0$. With the bounds for $A_\perp$ and $A_\perp^\dagger$, the first condition in (5.5) gives $|v(t)|/\gamma(t) \le \mathsf{D}^3 |v(t')|/\gamma(t')$ for its solutions at $a = 0$. By the bound for the first label derivatives of $\mathcal{M}_\perp$ and the choice of ℓ_*,

$$\sup_{t_0 \le t \le T} |\mathcal{M}_\perp(t, a) - \mathcal{M}_\perp(t, 0)| \le C\mathsf{D}^C |a| \le \mathsf{D}^{-3} \qquad (|a| \le 2\ell_*).$$

Let v solve the equation at such a label a on $[t', T]$. It solves the equation at the origin with the source $-(\mathcal{M}_\perp(\cdot, a) - \mathcal{M}_\perp(\cdot, 0))v$, so variation of constants gives

$$\frac{|v(t)|}{\gamma(t)} \le \mathsf{D}^3 \frac{|v(t')|}{\gamma(t')} + \int_{t'}^t \frac{|v(t'')|}{\gamma(t'')} \, \mathrm{d}t''.$$

The weights cancel because the bound at the origin is a ratio of values of γ. Since $t - t' \le T \le 1$, Gronwall's inequality gives $|v(t)|/\gamma(t) \le e\mathsf{D}^3|v(t')|/\gamma(t')$. For a solution of (5.18) on $[t_0, T]$, variation of constants therefore gives, at each phase and each label with $|a| \le 2\ell_*$,

$$\frac{|v(t,a)|}{\gamma(t)} \le e\mathsf{D}^3 \left(\frac{|v(t_0,a)|}{\gamma(t_0)} + \int_{t_0}^{t} \frac{|\psi(t',a)|}{\gamma(t')}\, dt' \right). \tag{5.23}$$

To compare the seed at a with its value at the origin, write $\mathbf{a} = A_\perp v_a$, with $v_a = A_\perp^\dagger \mathbf{a}$ at a. The difference $v_a - v_0$ solves the equation at the origin with the source $-(\mathcal{M}_\perp(\cdot,a) - \mathcal{M}_\perp(\cdot,0))v_a$. Its value at t_0 is $(A_\perp^\dagger(t_0,a) - A_\perp^\dagger(t_0,0))\mathbf{a}(t_0,0)$, by (5.4) and $A_\perp^\dagger P_\kappa = 0$, and $|\mathbf{a}(t_0,0)| = \gamma(t_0)\lambda^{-3/4}$. With the bound just proved for v_a, variation of constants gives $|v_a(t) - v_0(t)| \le C\mathsf{D}^C|a|\gamma(t)\lambda^{-3/4}$. Since $\gamma(t)\lambda^{-3/4} = |\mathbf{a}(t,0)|$, it follows that

$$|\mathbf{a}(t,a) - \mathbf{a}(t,0)| \le C\mathsf{D}^C|a|\,|\mathbf{a}(t,0)|, \qquad t_0 \le t \le T.$$

The label derivatives also give $|\kappa(t,a) - \kappa(t,0)| \le C\mathsf{D}^C|a||\kappa(t,0)|$ and $|M(t,a) - M(t,0)| \le C\mathsf{D}^C|a|$. Since $|M| \le \mathsf{D}$, the three bounds give

$$\big|(\kappa \cdot M\mathbf{a})(t,a) - (\kappa \cdot M\mathbf{a})(t,0)\big| \le C\mathsf{D}^C|a|\,|\kappa(t,0)|\,|\mathbf{a}(t,0)|,$$

while $(\kappa \cdot M\mathbf{a})(t,0) = \nu(t,0)|\kappa(t,0)||\mathbf{a}(t,0)| \ge \mathsf{D}^{-1}|\kappa(t,0)||\mathbf{a}(t,0)|$, by the second condition in (5.5). None of these exponents depends on the exponent of ℓ_*, so the choice of ℓ_* proves (5.19).

For (i), the data and source vanish outside the closed support of g. There the solution and all its label and phase derivatives vanish too. Taking the supremum in (5.23), with $T \le 1$, gives

$$\|v\|_{[0],\gamma} \le e\mathsf{D}^3\big(\gamma(t_0)^{-1}\|v(t_0)\|_{[0]} + \|\psi\|_{[0],\gamma}\big).$$

Here $\mathcal{M}_\perp$ enters through the identity map. Our choice of ℓ_* gives (D.7) for this coefficient. Lemma D.3 gives the same estimate at order n with constant $C_n\mathsf{D}^3$. The weight γ depends only on time. To return to the original coordinates, use

$$V = A_\perp v, \qquad v = A_\perp^\dagger V, \qquad \psi = A_\perp^\dagger f.$$

The product estimate (D.4) and Lemma D.2(c) bound each multiplication by $C_n\mathsf{D}^C$. This gives (5.20).

In (ii) the forcing vanishes for $t < t_0$, so $V(t_0) = 0$. Applying (5.20) to (5.17) bounds V. By (5.16), $\lambda\mathsf{p}$ is the phase primitive of a product of the coefficients κ/λ and $\lambda^2/|\kappa|^2$ with $f - 2MV$, so (D.5), (D.4) and (5.8) give

$$\lambda\|\mathsf{p}\|_{[n],\gamma} \le C_n\mathsf{D}^C\big(\|f\|_{[n],\gamma} + \|V\|_{[n],\gamma}\big), \tag{5.24}$$

which proves (5.21).

For (iii), the time independence of the envelope implies that $g\mathbf{a}$ solves the seed equation on $[t_0, T]$ at every label. It is perpendicular to κ and vanishes outside $\operatorname{supp} g$. By (5.4), its value at the time t_0 is $g(\mathrm{Id} - P_\kappa)\mathbf{a}(t_0,0)$. The chain rule gives $|g|_m \le C_m\ell_*^{-m}$, so $\|g\|_{[n]} \le C_n$. With (5.8) for P_κ and (D.4), this gives $\|g\mathbf{a}(t_0,\cdot)\|_{[n]} \le C_n\mathsf{D}^C|\mathbf{a}(t_0,0)| = C_n\mathsf{D}^C\gamma(t_0)\lambda^{-3/4}$. Hence (5.20) with $f = 0$ gives $\|g\mathbf{a}\|_{[n],\gamma} \le C_n\mathsf{D}^C\lambda^{-3/4}$. For a function of the phase alone, $\|F\|_{[n]} = \sum_{j\le n} h^j\|\partial_s^j F\|_\infty$, so Lemma 3.1 gives $\|F_h\|_{[n]} \le C_n h$. By (D.4), $\|A_1\|_{[n],\gamma} \le C_n\mathsf{D}^C h\lambda^{-3/4} = C_n\mathsf{D}^C\lambda^{-5/4}$. By (3.4), on $[t_0, T]$ the leading oscillation solves (5.15) with $f = 0$ and $\mathsf{p} = \mathsf{p}[A_1]$, so (5.24) gives (5.22). □

5.5. **The finite expansion.** Each level first determines a mean. Its transport of the leading wave enters the equation for the new oscillation. A curl then completes the oscillation to a divergence-free field. At depth P the perturbation is

$$W = \sum_{p=1}^{P} Z_p + C_{P+1}, \qquad Z_p = Y_p + B_p, \qquad Y_p = V_p + C_p, \qquad C_1 = 0,$$

where V_p is perpendicular to κ and B_p is independent of phase. The divergence correction $C_{p+1} = C[V_p]$ makes $V_p + C_{p+1} = \widetilde{\mathbf{d}} \times Q[V_p]$ divergence free by (3.6), which accounts for the last term C_{P+1}. We call p the *level* of Z_p and of Y_p, V_p, C_p and B_p. The factor $|\kappa|^{-1}$ in the vector potential gives the divergence correction C_p of V_{p-1} the bound of a term of level p, so we group it with V_p. The most expensive derivative comes from differentiating the phase in physical space. It costs $\mathsf{D}^2\lambda^{3/2}$ at most, but its coefficient can only be a mean or a divergence correction, since $V_i \cdot \kappa = 0$. The smallest resulting gain is $\lambda^{-3/4}$ up to parent factors. Keeping $\lambda^{-2/3}$ per level leaves $\lambda^{-1/12}$ to absorb them.

5.5.1. *The equations at each level.* Start from V_1, p_1, C_2 in (3.7) and put $B_1 = 0$, so that $Y_1 = Z_1 = V_1$. Set aside the activation error $R_{\widetilde{\chi}}$ of (3.8). The terms left by the leading wave at level two are

$$H_2 = \widetilde{\chi}\big[(\partial_t + M)C[A_1] + \mathbf{d}\mathsf{p}[A_1]\big] + (V_1 \cdot \mathbf{d})V_1 + (C_2 \cdot \kappa)\partial_s V_1. \tag{5.25}$$

The first mean is forced by a slow derivative of an averaged stress. The linear term has zero phase mean. In the last term we can transfer the fast derivative to the divergence correction. Indeed, (3.6) and integration over one period give

$$\langle (C_2 \cdot \kappa)\partial_s V_1 \rangle_s = -\langle \partial_s (C_2 \cdot \kappa) V_1 \rangle_s = \langle (\mathbf{d} \cdot V_1) V_1 \rangle_s.$$

Adding the other quadratic term gives the row divergence of $\langle V_1 \otimes V_1 \rangle_s$, as in the mean equation of Cheverry [34, Section 4.1, (4.19)]. In components, $(\langle H_2 \rangle_s)_\ell = \sum_i \mathbf{d}_i \langle (V_1)_\ell (V_1)_i \rangle_s$. Both factors have the size of V_1. The remaining derivative acts at fixed phase, so it costs only a power of D. By (5.22) and (D.3), $\|\langle H_2 \rangle_s\|_{[n]} \le C_n \mathsf{D}^C \lambda^{-5/2}$. The residual $R_U \circ X$ has the same bound by (5.10). We include it in the first mean equation.

At level p we collect the label-advection terms with $i + j = p$ and the phase-advection terms with $i + j = p + 1$. Only C_i and B_i can advect the phase, since $V_i \cdot \kappa = 0$. We set aside the terms $(B_i \cdot \kappa)\partial_s Y_j$ until we have found the mean at this level. The other terms give, for $p \ge 3$,

$$\begin{aligned} H_p = {} & (\partial_t + M)C_p + \mathbf{d}\mathsf{p}_{p-1} \\ & + \sum_{i+j=p} (Z_i \cdot \mathbf{d})Z_j + \sum_{\substack{i+j=p+1 \\ i \ge 2}} (C_i \cdot \kappa)\partial_s Y_j, \end{aligned} \tag{5.26}$$

where all summation indices are positive integers. For every $p \ge 2$ put

$$f_p = -(H_p - \langle H_p \rangle_s), \qquad \bar{f}_p = -\langle H_p \rangle_s \quad (p \ge 3), \qquad \bar{f}_2 = -\langle H_2 \rangle_s - R_U \circ X. \tag{5.27}$$

Every field in H_p is now known. In particular the level-p correction $C_p = C[V_{p-1}]$ was computed from the preceding oscillation. Solve the mean equation (5.11) on $[0, T]$ for $B_p, \bar{q}_p$, then put

$$f_p^S = -\sum_{i=2}^{p} (B_i \cdot \kappa)\partial_s Y_{p+1-i}, \tag{5.28}$$

and apply (5.15) to $f_p + f_p^S$, obtaining V_p, p_p. Finally set $Q_p = Q[V_p]$ and $C_{p+1} = C[V_p]$. Each summand in (5.28) has zero mean in s. The summand with $i = p$ is the transport of V_1 by B_p.

Reflection of both a and s changes the sign of $A_1 = g\mathbf{a}F_h$ and of $R_U \circ X$. The coefficients $D_a X$, M and κ are even. Each spatial derivative reverses parity, so the quadratic advection of odd velocities is odd. The mean and transverse equations, with their respective endpoint and initial conditions, preserve this parity by uniqueness. The zero-mean phase primitive in $Q[V_p]$ is even and its label curl C_{p+1} is odd. Induction gives $W(t, -a, -s) = -W(t, a, s)$. Since X is odd, evaluation at $s = k_* \cdot a$ gives an odd Eulerian velocity.

Proposition 5.4. *For every integer $P \geq 2$, the profiles obtained from (5.11), (5.27) and (5.28) are well defined, and*

$$W = \sum_{p=1}^{P} Z_p + C_{P+1} = \sum_{p=1}^{P} (V_p + C_{p+1} + B_p), \qquad \mathsf{P} = \sum_{p=1}^{P} (\mathsf{p}_p + \bar{q}_p), \quad \bar{q}_1 = 0,$$

satisfy $\widetilde{\mathbf{d}} \cdot W = 0$ and

$$W_t + MW + (W \cdot \widetilde{\mathbf{d}})W + \widetilde{\mathbf{d}}\mathsf{P} + R_U \circ X = R_P + R_{\widetilde{\chi}}. \tag{5.29}$$

Since $\langle V_p \rangle_s = \langle C_p \rangle_s = 0$, the oscillatory part $W_{\mathrm{osc}} = W - \langle W \rangle_s = \sum_{p=1}^{P} (V_p + C_{p+1})$ is supported in $\operatorname{supp} g$, and so are the sources $f_p + f_p^S$. The fields V_p, C_p, p_p and $f_p + f_p^S$ vanish for $t < t_0$. Set $V_{P+1} = B_{P+1} = 0$, so that $Y_{P+1} = Z_{P+1} = C_{P+1}$. The remainder of the expansion is

$$\begin{aligned} R_P = {} & (\partial_t + M) C_{P+1} + \mathbf{d}\mathsf{p}_P \\ & + \sum_{\substack{i,j \leq P+1 \\ i+j \geq P+1}} (Z_i \cdot \mathbf{d}) Z_j + \sum_{\substack{i,j \leq P+1 \\ i+j \geq P+2}} (Z_i \cdot \kappa) \partial_s Y_j. \end{aligned} \tag{5.30}$$

At the labels outside $\operatorname{supp} g$ it is independent of the phase. The construction is jointly smooth.

Proof. By (3.6) and (5.11),

$$\widetilde{\mathbf{d}} \cdot W = \sum_{p=1}^{P} \big[\widetilde{\mathbf{d}} \cdot (\widetilde{\mathbf{d}} \times Q_p) + \mathbf{d} \cdot B_p \big] = 0.$$

With $W = \sum_{i=1}^{P+1} Z_i$, the quadratic term is

$$(W \cdot \widetilde{\mathbf{d}})W = \sum_{i,j} (Z_i \cdot \mathbf{d}) Z_j + \sum_{i,j} ((C_i + B_i) \cdot \kappa) \partial_s Y_j$$

with $1 \leq i, j \leq P+1$. For $2 \leq p \leq P$, the terms to be canceled at level p sum to $H_p - f_p^S$, together with $R_U \circ X$ at $p = 2$. The new linear terms

$$(\partial_t + M)(V_p + B_p) + \kappa \partial_s \mathsf{p}_p + \mathbf{d}\bar{q}_p = f_p + f_p^S + \bar{f}_p$$

cancel them, since the right side is $-H_p + f_p^S$ for $p \geq 3$ and $-H_2 + f_2^S - R_U \circ X$ for $p = 2$. For V_1, (3.4) and (3.7) give

$$V_{1,t} + MV_1 + \kappa \partial_s \mathsf{p}_1 = \widetilde{\chi}' A_1, \qquad (\partial_t + M) C_2 + \mathbf{d}\mathsf{p}_1 = \widetilde{\chi} \big[(\partial_t + M) C[A_1] + \mathbf{d}\mathsf{p}[A_1] \big] + \widetilde{\chi}' C[A_1].$$

The two terms with $\widetilde{\chi}'$ sum to $R_{\widetilde{\chi}}$, and the remaining term is canceled in H_2. The surviving products have $i + j > P$ for $\mathbf{d}$ and $i + j - 1 > P$ for $\kappa \partial_s$. Together with the two linear terms left at level $P + 1$, these give (5.30).

Smoothness follows by induction from the transverse ordinary differential equation, Theorem 4.2 for the mean, and the curl formula. For support, let $\mathcal{O}$ consist of the pairs (t, a) with $t < t_0$ or $a \notin \operatorname{supp} g$. There V_1, C_2, p_1 vanish. If the oscillatory fields of all lower levels vanish on $\mathcal{O}$, H_p there contains only $(B_i \cdot \mathbf{d})B_j$. Its oscillatory part and the source f_p^S vanish. The zero initial data in (5.17) give $V_p = 0$ there, hence also its curl correction and the zero-mean pressure primitive (5.16). This proves the support assertion by induction. On $\mathcal{O}$ the remainder (5.30) likewise contains only mean products. The means themselves may extend to time zero. □

5.5.2. *Bounds at every order.* To bound the linear terms in (5.25) and (5.26), we use the transverse equation to evaluate the time derivative of the curl correction.

Lemma 5.5. *There is an absolute exponent C such that, for every $n \geq 0$ and every field V with $\langle V\rangle_s = 0$ and $\kappa \cdot V = 0$,*

$$\|C[V]\|_{[n],\gamma} \leq C_n \mathsf{D}^C \lambda^{-1} \|V\|_{[n+1],\gamma}. \tag{5.31}$$

If moreover $V_t + MV + \kappa \partial_s \mathsf{p} = f$ with $\langle f\rangle_s = 0$ and $\langle \mathsf{p}\rangle_s = 0$, then

$$\|(\partial_t + M)C[V] + \mathbf{d}\mathsf{p}\|_{[n],\gamma} \leq C_n \mathsf{D}^C \lambda^{-1} \big(\|V\|_{[n+1],\gamma} + \|f\|_{[n+1],\gamma}\big). \tag{5.32}$$

For the leading oscillation $V = A_1$ on $[t_0, T]$, the source in this estimate is $f = 0$.

Proof. By (3.5), $Q[V]$ is λ^{-1} times the phase primitive of V, multiplied by the coefficients κ/λ and $\lambda^2/|\kappa|^2$, and $C[V] = \mathbf{d} \times Q[V]$. Since ℓ_*^{-1} is a power of D, (D.4), (D.3), (D.5) and (5.8) give (5.31).

For (5.32) use $\kappa_t = -M^T \kappa$, $(|\kappa|^{-2})_t = 2(\kappa \cdot M\kappa)|\kappa|^{-4}$, and $[\partial_t, \mathbf{d}_j] = -M_{\ell j}\mathbf{d}_\ell$. Differentiating $Q[V]$ in time and inserting the equation for V_t, in which the term $\kappa \times \kappa\, \partial_s \mathsf{p}$ vanishes, gives

$$\partial_t Q[V] = \partial_s^{-1} \left[\frac{(M^T\kappa) \times V + \kappa \times MV - \kappa \times f}{|\kappa|^2} - 2\frac{(\kappa \times V)(\kappa \cdot M\kappa)}{|\kappa|^4} \right]. \tag{5.33}$$

Commuting ∂_t past the curl gives $(\partial_t + M)C[V] = \mathbf{d} \times \partial_t Q[V] + [\partial_t, \mathbf{d}\times]Q[V] + MC[V]$. By (5.33) and the formula for $\partial_s \mathsf{p}$ in (5.16), each term, including $\mathbf{d}\mathsf{p}$, carries a factor λ^{-1} and at most one label derivative. The product and primitive estimates therefore give (5.32) without a time derivative of M, f or a mean. The weight γ is applied after the time differentiation. □

We can now compute the gain per level. Suppress fixed powers of D. The leading oscillation has size $\lambda^{-5/4}$ by (5.22). Its curl correction costs a label derivative and gains λ^{-1} by (5.31). When this correction advects the phase, its size $\lambda^{-9/4}$ multiplies the fast derivative cost $\lambda^{3/2}$, leaving $\lambda^{-3/4}$. The first mean, of size $\lambda^{-5/2}$ by (5.25) and (5.12), leaves λ^{-1} on the same calculation. Advection in the labels leaves $\lambda^{-5/4}$. Thus even the first of these three terms has $\lambda^{-1/12}$ to spare beyond the gain $\lambda^{-2/3}$.

At each level we prove the estimates at every derivative order. A derivative lost in the source bounds can therefore be taken at a previous level. The new mean B_p enters (5.28) as a multiplier of $\partial_s V_1$, so its order-n estimate suffices for the order-n transverse source. For each wave j the expansion stops at its chosen depth P. We take the limit in j of these completed wave additions. At any one step, only finitely many level constants $C_{n,p}$ through $p = P$ enter the estimates at the prescribed derivative order q_j. The frequency schedule absorbs these constants. An additional power of D at each level would also raise the frequency threshold with the depth. The induction in (5.40) avoids that loss.

Theorem 5.6. *There is an absolute exponent C such that the following holds if $\mathsf{D}^C \lambda^{-1/12} \leq 1$. For all integers $n \geq 0$ and $p \geq 2$, the construction satisfies*

$$\|V_p\|_{[n],\gamma} + \|f_p + f_p^S\|_{[n],\gamma} + \lambda \|\mathsf{p}_p\|_{[n],\gamma} \leq C_{n,p} \mathsf{D}^C \lambda^{-5/4} \lambda^{-2(p-1)/3}, \tag{5.34}$$

$$\|B_p\|_{[n]} + \|\bar{f}_p\|_{[n]} \leq C_{n,p} \mathsf{D}^C \lambda^{-5/2} \lambda^{-2(p-2)/3}, \tag{5.35}$$

$$\|C_{p+1}\|_{[n],\gamma} \leq C_{n,p} \mathsf{D}^C \lambda^{-9/4} \lambda^{-2(p-1)/3}. \tag{5.36}$$

The bounds (5.34) and (5.36) also hold for $p = 1$, with $f_1 = f_1^S = 0$. For all integers $P \geq 2$ and $m \geq 0$, the expansion of Proposition 5.4 at the depth P satisfies

$$\|W_{\mathrm{osc}}\|_{[m],\gamma} \leq C_{m,P} \mathsf{D}^C \lambda^{-5/4}, \qquad \Big\| \sum_{p=2}^{P} B_p \Big\|_{[m]} + \Big\| \sum_{p=2}^{P} \bar{f}_p \Big\|_{[m]} \leq C_{m,P} \mathsf{D}^C \lambda^{-5/2}, \tag{5.37}$$

and its remainder satisfies

$$\|R_P\|_{[m]} \le C_{m,P}\lambda^{-5/4}\lambda^{-2P/3}. \tag{5.38}$$

Proof. By (D.4), (D.3) and (5.8), since ℓ_*^{-1} is a power of D and $\kappa\partial_s$ costs $|\kappa|h^{-1} \le \mathsf{D}^2\lambda^{3/2}$ by (5.3), fields Z, Z', Y satisfy, with the weight on either factor of a product or on neither,

$$\begin{aligned} \|(Z\cdot\mathbf{d})Z'\|_{[n],\gamma} + \|(\mathbf{d}\cdot Z)Z'\|_{[n],\gamma} &\le C_n\mathsf{D}^C\|Z\|_{[n+1]}\|Z'\|_{[n+1],\gamma}, \\ \|(Z\cdot\kappa)\partial_s Y\|_{[n],\gamma} &\le C_n\mathsf{D}^C\lambda^{3/2}\|Z\|_{[n]}\|Y\|_{[n+1],\gamma}. \end{aligned} \tag{5.39}$$

For $p = 1$, (5.22) and (5.31) give (5.34) and (5.36) for A_1, $\mathsf{p}[A_1]$, $C[A_1]$ on $[t_0, T]$, and $0 \le \widetilde{\chi} \le 1$ gives them for V_1, p_1, C_2.

To keep powers of D from accumulating with the depth, we propagate a bound for the transverse source and the mean that contains only powers of λ. For $p \ge 2$ the induction claim at every order is

$$\|f_p + f_p^S\|_{[n],\gamma} + \lambda^{1/2}\|B_p\|_{[n]} \le C_{n,p}\lambda^{-5/4}\lambda^{-2(p-1)/3}. \tag{5.40}$$

For B_p this is (5.35) with $\lambda^{1/12}$ in place of D^C. Since (5.40) contains no power of D, every exponent of D below is a sum of a bounded number of the exponents of (5.12), (5.21), (5.22), (5.31), (5.32) and (5.39), and it depends neither on the level nor on the depth. The theorem holds with the largest of these exponents, since $\mathsf{D} \ge 2$. In the rest of the proof every bound holds up to a factor $C_{n,p}$ or $C_{m,P}$, which we do not write.

The bound (5.40) at a level $i \ge 2$ gives (5.34) and (5.36) for V_i, p_i and C_{i+1}. Indeed, by Proposition 5.4 the source $f_i + f_i^S$ vanishes for $t < t_0$ and outside $\operatorname{supp} g$, so (5.21) bounds V_i and $\lambda\mathsf{p}_i$ by D^C times the right side of (5.40), and (5.31) then bounds C_{i+1}. For the bounds at each level it remains to prove (5.40) and (5.35).

Let $p \ge 2$, and suppose that (5.40) holds at the levels $2 \le i < p$ and at every order. Then (5.34) and (5.36) hold for V_i and C_{i+1}, $1 \le i < p$. Recall $Y_i = V_i + C_i$ and $Z_i = Y_i + B_i$. The right side of (5.36) for C_i and the bound (5.40) for B_i are at most the right side of (5.34) for V_i. Since $\gamma \le 1$, it follows that, for $i < p$,

$$\|Y_i\|_{[n],\gamma} + \|Z_i\|_{[n]} \le \mathsf{D}^C\lambda^{-5/4}\lambda^{-2(i-1)/3}. \tag{5.41}$$

The source f_p in (5.27) uses only the fields V_i, p_i and B_i with $i < p$, and the corrections C_i with $i \le p$. Subtracting the phase mean of $(Z_i\cdot\mathbf{d})Z_j$ removes $(B_i\cdot\mathbf{d})B_j$, and every other product has a factor Y_i or Y_j, to which we apply the γ norm. By (5.32) and (5.34) for V_{p-1}, the linear term of H_p is at most $\mathsf{D}^C\lambda^{-9/4}\lambda^{-2(p-2)/3}$; for $p = 2$ this is the bound for A_1, with $f = 0$ and $\widetilde{\chi} \le 1$. By (5.39) and (5.41), a product $(Z_i\cdot\mathbf{d})Z_j$ with $i + j = p$ is at most $\mathsf{D}^C\lambda^{-5/2}\lambda^{-2(p-2)/3}$. By (5.39), (5.41) and (5.36) for C_i, a product $(C_i\cdot\kappa)\partial_s Y_j$ with $i + j = p + 1$ is at most $\mathsf{D}^C\lambda^{3/2}\lambda^{-9/4}\lambda^{-5/4}\lambda^{-2(p-2)/3} = \mathsf{D}^C\lambda^{-2}\lambda^{-2(p-2)/3}$. The right side of (5.40) is $\lambda^{-23/12}\lambda^{-2(p-2)/3}$, so the three bounds are this number times

$$\mathsf{D}^C\lambda^{-1/3}, \qquad \mathsf{D}^C\lambda^{-7/12}, \qquad \mathsf{D}^C\lambda^{-1/12},$$

respectively. Thus $\|f_p\|_{[n],\gamma}$ is at most $\mathsf{D}^C\lambda^{-1/12}$ times the right side of (5.40).

In the mean source $\bar{f}_p$, integration by parts in s removes the fast derivative just as at level two. By (3.6), for $i \ge 2$ we have

$$\langle (C_i\cdot\kappa)\partial_s Y_j\rangle_s = \langle (\mathbf{d}\cdot V_{i-1})Y_j\rangle_s. \tag{5.42}$$

The linear terms have zero phase average because C_p and p_{p-1} do. Products of one mean and one oscillation also average to zero, since B_i and κ are phase independent and $\langle Y_j\rangle_s = 0$. Thus, for

$p \geq 2$, the mean source is

$$-\bar{f}_p = \sum_{i+j=p} \langle (Y_i \cdot \mathbf{d}) Y_j \rangle_s + \sum_{i+j=p} \langle (\mathbf{d} \cdot V_i) Y_j \rangle_s + \sum_{i+j=p} (B_i \cdot \mathbf{d}) B_j + \mathbf{1}_{\{p=2\}} R_U \circ X,$$

where the last term occurs only at $p = 2$. By (5.39) and (5.41), the first two sums are at most $\mathsf{D}^C \lambda^{-5/2} \lambda^{-2(p-2)/3}$. By (5.39) and (5.40) at the lower levels, the third is at most $\mathsf{D}^C \lambda^{-7/2} \lambda^{-2(p-2)/3}$, which is smaller. The term $R_U \circ X$ is at most $C_n \lambda^{-5/2}$, by (5.10). Hence $\|\bar{f}_p\|_{[n]} \leq \mathsf{D}^C \lambda^{-5/2} \lambda^{-2(p-2)/3}$ at every order. With this estimate at the order $n+1$, (5.12) bounds $\|B_p\|_{[n]}$ by D^C times as much. This proves (5.35) at the level p, and with the hypothesis $\mathsf{D}^C \leq \lambda^{1/12}$ it gives (5.40) for B_p.

For f_p^S in (5.28), the means B_i, $2 \leq i \leq p$, are now bounded by (5.40), and Y_{p+1-i} is bounded by (5.41). By (5.39), each product $(B_i \cdot \kappa) \partial_s Y_{p+1-i}$ is therefore at most $\mathsf{D}^C \lambda^{-3/2} \lambda^{-2(p-1)/3}$, which is $\mathsf{D}^C \lambda^{-1/4}$ times the right side of (5.40). With the bound for f_p, $\|f_p + f_p^S\|_{[n],\gamma}$ is at most $\mathsf{D}^C \lambda^{-1/12}$ times that right side, and the hypothesis gives (5.40) at the level p. This completes the induction. The number of products in a level and the constants of the lower levels at higher orders enter $C_{n,p}$.

Since $\lambda \geq 1$, summing (5.34)–(5.36) over $p \leq P$ gives (5.37). For the remainder, recall $Y_{P+1} = Z_{P+1} = C_{P+1}$. By (5.36) for C_{P+1}, the bound (5.41) holds for $i \leq P+1$. The products in (5.30) are therefore bounded as those of H_p and f_p^S were: the products $(Z_i \cdot \mathbf{d}) Z_j$ with $i + j = q$, and $(C_i \cdot \kappa) \partial_s Y_j$ and $(B_i \cdot \kappa) \partial_s Y_j$ with $i + j = q + 1$, are at most $\mathsf{D}^C \lambda^{-1/12}$ times $\lambda^{-5/4} \lambda^{-2(q-1)/3}$, and in (5.30) they have $q \geq P + 1$. By (5.32) and (5.34) for V_P, the linear term $(\partial_t + M) C_{P+1} + \mathbf{d}\mathsf{p}_P$ is at most $\mathsf{D}^C \lambda^{-1/3}$ times $\lambda^{-5/4} \lambda^{-2P/3}$. The products of two means are bounded in the unweighted norm, and every other term in the weighted norm, which dominates the unweighted one because $\gamma \leq 1$. These other terms vanish for $t < t_0$ by Proposition 5.4, so the comparison holds on all of $[0, T]$. With the hypothesis, this is (5.38). □

5.6. **The approximate solution.** Fix an integer $P \geq 2$. The profiles W, P of Proposition 5.4 at depth P give the Eulerian velocity and pressure

$$V = U + W^\natural, \qquad \Pi = p_U + \mathsf{P}^\natural, \qquad W = W_{\mathrm{osc}} + \bar{W}, \quad \bar{W} = \sum_{p=2}^{P} B_p.$$

Phase averaging precedes evaluation at $s = k_* \cdot a$. The evaluated pressure increment can therefore have a spatial mean. Since p_U has mean zero, the normalized pressure is $\Pi(t,x) - \int_{\mathbb{T}^3} \mathsf{P}^\natural(t,y)\, \mathrm{d}y$. Its spatial derivatives agree with those of Π. The momentum residual R_V can have nonzero divergence, which enters the pressure equation as $\Delta \Pi = \operatorname{div} R_V - \operatorname{tr}((\nabla V)^2)$.

Fix an absolute integer $b \geq 1$, at least as large as the exponents C of Theorem 5.6 and of Lemma 5.7, and sufficiently large for the finitely many estimates of this subsection and the next. Put

$$C_{\mathrm{step}} = \Gamma_P \mathsf{D}^b,$$

where $\Gamma_P \geq 1$ depends only on P and the fixed profiles. We take it sufficiently large compared with every $C_{2,P}$ in this subsection and the next. The estimates of the residual, the pressure Hessian and the flow use the profile bounds through order two. The truncation uses the velocity bound in $C^{P,1/2}$ and the tail estimate (B.2). From here on we assume

$$C_{\mathrm{step}} \lambda^{-1/12} \leq 1. \tag{5.43}$$

Since $C_{\mathrm{step}} \geq \mathsf{D}^C$ for the exponent C of Theorem 5.6, the hypothesis of that theorem holds.

To estimate the velocity and residual in Eulerian coordinates, we evaluate each profile at $a = X(t,\cdot)^{-1}(x)$ and $s = k_* \cdot a$.

Lemma 5.7. *Let $g_0(t,a,s)$ be smooth and independent of the phase at the labels outside* $\operatorname{supp} g$. *Assume its phase average extends smoothly and periodically in a across the cell faces. There is an absolute exponent C such that, for every integer $m \geq 0$ and every $t \in [t_0, T]$,*

$$\sum_{|\beta|=m} \|\partial_x^\beta g_0^\natural(t)\|_{L^\infty} \leq C_m(\mathsf{D}^C\lambda^{3/2})^m\gamma(t)\|g_0\|_{[m],\gamma}, \tag{5.44}$$

$$\|g_0^\natural(t)\|_{C^{m,1/2}} \leq C_m(\mathsf{D}^C\lambda^{3/2})^{m+1}\gamma(t)\|g_0\|_{[m+1],\gamma}. \tag{5.45}$$

Both bounds also hold on $[0,T]$ with γ replaced by one, and for a phase-independent field with $\mathsf{D}^C\lambda^{3/2}$ replaced by D^C.

Proof. Since $\operatorname{supp} g$ lies in the interior of the cell, g_0 equals its phase average near every cell face, and the assumed smooth periodic continuation of that average continues $g_0(t,a,k_*\cdot a)$ across the faces for every k_*. The field $\widetilde{\mathbf{d}}_i g_0$ is again independent of the phase outside $\operatorname{supp} g$, so, after composition with the inverse flow, $g_0^\natural$ and its derivatives extend smoothly and periodically. The spatial chain rule in (3.1) and $\widetilde{\mathbf{d}} = (D_aX)^{-T}(\nabla_a + k_*\partial_s)$ give

$$\sum_i \|\widetilde{\mathbf{d}}_i g_0\|_{[n],\gamma} \leq C_n\mathsf{D}^C\lambda^{3/2}\|g_0\|_{[n+1],\gamma}, \qquad n \geq 0,$$

by Lemma D.1 and (5.8): a label derivative costs ℓ_*^{-1}, a fixed power of D, and $k_*\partial_s$ costs $|k_*|h^{-1} \leq \mathsf{D}\lambda^{3/2}$, by (5.3). Iterate the displayed bound; at each time the supremum of a field is at most $\gamma(t)$ times its weighted norm of order zero. This proves (5.44). The bound $\|v\|_{C^{0,1/2}} \leq C(\|v\|_\infty + \|\nabla v\|_\infty)$, applied to derivatives through order m, proves (5.45) with one additional derivative. A phase-independent field has $\widetilde{\mathbf{d}}g_0 = \mathbf{d}g_0$, proving the last assertion. □

The identity of Proposition 5.4 now gives the physical residual. The phase evaluation is significant in its estimate. To control one Eulerian derivative in $C^{0,1/2}$ we take two spatial derivatives in the supremum norm. They can cost λ^3 up to a power of D, changing the profile bound $\lambda^{-5/4-2P/3}$ into $\lambda^{7/4-2P/3}$.

Lemma 5.8. *The field V is smooth, odd and divergence free, Π is smooth, and*

$$V_t + V\cdot\nabla V + \nabla\Pi = R_V, \qquad R_V = R_P^\natural + R_{\widetilde{\chi}}^\natural. \tag{5.46}$$

The oscillation satisfies

$$\|W_{\mathrm{osc}}^\natural\|_{C_tL^\infty} \leq C_{0,P}\mathsf{D}^b\lambda^{-5/4}, \tag{5.47}$$

and for every $m \geq 0$,

$$\begin{gathered}\|W_{\mathrm{osc}}^\natural\|_{C_tC^{m,1/2}} \leq C_{m,P}\mathsf{D}^{b(m+2)}\lambda^{3(m+1)/2-5/4}, \qquad \|\bar{W}^\natural\|_{C_tC^{m,1/2}} \leq C_{m,P}\mathsf{D}^{b(m+2)}\lambda^{-5/2},\\ \|R_V\|_{C_tC^{1,1/2}} \leq C_{\mathrm{step}}\lambda^2\big(\lambda^{-2P/3} + \|\widetilde{\chi}'\gamma\|_\infty\big),\end{gathered} \tag{5.48}$$

where the last supremum is over $[t_0,T]$.

Proof. By (3.1), at $x = X(t,a)$ one has $(\partial_t + U\cdot\nabla)W^\natural = (W_t)^\natural$, $W^\natural\cdot\nabla U = (MW)^\natural$, $W^\natural\cdot\nabla W^\natural = ((W\cdot\widetilde{\mathbf{d}})W)^\natural$ and $\nabla\mathsf{P}^\natural = (\widetilde{\mathbf{d}}\mathsf{P})^\natural$. Adding the definition of R_U and (5.29) gives (5.46), because $(R_U\circ X)^\natural = R_U$: the residual of the parent is canceled exactly. Also $\widetilde{\mathbf{d}}\cdot W = 0$ gives $\operatorname{div} V = 0$. Since $\operatorname{supp} g$ lies in the interior of the cell, the oscillatory terms in $W^\natural$ and $\mathsf{P}^\natural$ extend smoothly and periodically for every k_*. Proposition 5.4 gives the smoothness of V and Π, and $W^\natural$ is odd by the symmetry of the construction of Subsection 5.5 under $(a,s)\mapsto(-a,-s)$.

Before t_0 the oscillation vanishes, and afterward $\gamma \leq 1$. Thus (5.37) bounds its global supremum by (5.47). Apply Lemma 5.7 to the same profile bound at order $m+1$, using the phase-independent estimate for $\bar{W}$. This gives both velocity estimates in (5.48). For R_P the unweighted estimate at $m=1$ and (5.38) give $\|R_P^\natural\|_{C_tC^{1,1/2}} \leq C_{2,P}\mathsf{D}^C\lambda^{7/4-2P/3}$. For $R_{\widetilde{\chi}} = \widetilde{\chi}'(A_1 + C[A_1])$, use (5.22) and

(5.31) in the weighted estimate at each t. This gives $C\mathsf{D}^C\lambda^{7/4}|\widetilde{\chi}'(t)|\gamma(t)$ in $C^{1,1/2}$. Adding proves the residual estimate. □

The same slope $F_h'(0) = 1$ produces the central shear and the leading pressure curvature. Its contribution to the pressure Hessian is negative where the slope is positive, because $\kappa \cdot M\mathbf{a} > 0$ on the envelope. Where the slope is negative, its size is only $O(h)$. This is the upper bound needed in the displacement form (4.7). We estimate it across the torus while retaining the leading velocity gradient at the origin.

Lemma 5.9. *On $[0,T]\times\mathbb{T}^3$, in the sense of quadratic forms,*

$$D_x^2\mathsf{P}^\natural \le C_{\mathrm{step}}\lambda^{-1/4}\,\mathrm{Id}\,. \tag{5.49}$$

At $x = 0$, for $t_0 \le t \le T$,

$$\nabla V(t,0) = \nabla U(t,0) + \widetilde{\chi}(t)\,\mathbf{a}(t,0)\otimes\kappa(t,0) + E_c(t), \qquad |E_c| \le \lambda^{-1/3}. \tag{5.50}$$

Proof. Write the leading pressure as $\mathsf{p}_1 = -c_{\mathbf{a}}P_h$, with $c_{\mathbf{a}} = 2\widetilde{\chi}g(\kappa\cdot M\mathbf{a})/|\kappa|^2$. Since $D_x^2\mathsf{p}_1^\natural = (\widetilde{\mathbf{d}}\,\widetilde{\mathbf{d}}\,\mathsf{p}_1)^\natural$ and $\mathbf{d}_i\kappa_j = \mathbf{d}_j\kappa_i$, with $a = X(t,\cdot)^{-1}(x)$ and $s = k_*\cdot a$ on the right,

$$D_x^2\mathsf{p}_1^\natural = -c_{\mathbf{a}}F_h'\kappa\otimes\kappa + (\mathbf{d}\partial_s\mathsf{p}_1)\otimes\kappa + \kappa\otimes(\mathbf{d}\partial_s\mathsf{p}_1) + (\partial_s\mathsf{p}_1)\mathbf{d}\kappa + \mathbf{d}^2\mathsf{p}_1. \tag{5.51}$$

On $[t_0,T]\times\operatorname{supp} g$, (5.19) gives $\kappa\cdot M\mathbf{a} > 0$. Outside this set $\widetilde{\chi}g = 0$. Since $0 \le \widetilde{\chi}g \le 1$ everywhere, it follows that $0 \le c_{\mathbf{a}}|\kappa|^2 \le 2|\kappa||M||\mathbf{a}|$. With $F_h' \ge -Ch$, the first tensor on the right is therefore at most $Chc_{\mathbf{a}}|\kappa|^2\,\mathrm{Id}$. On $[t_0,T]\times\operatorname{supp} g$, (5.19) gives $|\mathbf{a}| \le 2\lambda^{-3/4}$, while $|\kappa| \le \mathsf{D}^2\lambda$ and $|M| \le \mathsf{D}$ by (5.3) and (5.1). So $c_{\mathbf{a}}|\kappa|^2 \le 4\mathsf{D}^3\lambda^{1/4}$ and, since $h = \lambda^{-1/2}$, the first tensor is at most $C\mathsf{D}^3\lambda^{-1/4}\,\mathrm{Id}$. By (D.4), (D.3) and (5.8), the sum of the remaining tensors is at most $C\mathsf{D}^C\lambda^{3/2}\|\mathsf{p}_1\|_{[2]}$ in supremum norm, and $\lambda\|\mathsf{p}_1\|_{[2]} \le C\mathsf{D}^C\lambda^{-5/4}$ by (5.22). So their sum is at most $C\mathsf{D}^C\lambda^{-3/4}$.

For the higher oscillatory pressures, (5.34) gives the least decay in λ at $p = 2$, where it bounds p_2 by $C_{2,P}\mathsf{D}^C\lambda^{-1-23/12}$. Two Eulerian derivatives in (5.44) would cost λ^3, leaving $\lambda^{1/12}$. To bound the first derivative, use the transverse equation (5.16). For $2 \le p \le P$ it gives

$$\nabla_x\mathsf{p}_p^\natural = (\mathbf{d}\mathsf{p}_p + \kappa\partial_s\mathsf{p}_p)^\natural, \qquad \kappa\partial_s\mathsf{p}_p = P_\kappa(f_p + f_p^S - 2MV_p).$$

The label derivative costs a power of D, and the phase term is bounded by the source and velocity in (5.34). The coefficient estimates therefore give

$$\Big\|\widetilde{\mathbf{d}}\sum_{p=2}^{P}\mathsf{p}_p\Big\|_{[1],\gamma} \le C_{2,P}\mathsf{D}^C\lambda^{-23/12}.$$

The oscillatory pressures are supported in $\operatorname{supp} g$ in Lagrangian coordinates. One further Eulerian derivative costs $\lambda^{3/2}$ up to a power of D, so (5.44) gives

$$\Big\|D_x^2\sum_{p=2}^{P}\mathsf{p}_p^\natural\Big\|_\infty \le C_{2,P}\mathsf{D}^C\lambda^{-5/12}. \tag{5.52}$$

For the mean pressures, add the mean equations (5.11) of the levels $2 \le p \le P$. In Eulerian coordinates the sum is the linearized Euler equation (4.12) for $\bar{W}^\natural$ over U, with the pressure $\sum_p \bar{q}_p^\natural$ and the source $(\sum_p \bar{f}_p)^\natural$, and its divergence is

$$\Delta\sum_{p=2}^{P}\bar{q}_p^\natural = \operatorname{div}\Big(\sum_{p=2}^{P}\bar{f}_p\Big)^\natural - 2\operatorname{tr}\big(\nabla U\,\nabla\bar{W}^\natural\big).$$

The operator $D^2\Delta^{-1}$ is bounded on $C^{0,1/2}$. By (5.45) for phase-independent fields and (5.37), the source and $\bar{W}^\natural$ are at most $C_{2,P}\mathsf{D}^C\lambda^{-5/2}$ in $C_tC^{1,1/2}$, and $\|\nabla U\|_{C_tC^{0,1/2}} \le C\mathsf{D}\,\|\nabla U\|_\infty \le C\mathsf{D}^2$ by (B.1) and (5.1). Hence the Hessian of the sum of the mean pressures is at most $C_{2,P}\mathsf{D}^C\lambda^{-5/2}$ in

supremum norm. Together with the bounds for the tensors of (5.51) and with (5.52), this proves (5.49).

To compute the velocity gradient at $x = 0$, evaluate at $a = 0$ and $s = 0$, since the flow of U fixes the origin. As $g(0) = F_h'(0) = 1$ and $F_h(0) = 0$, the chain rule (3.1) gives $D_x V_1^\natural(t,0) = \widetilde{\chi}(t)\mathbf{a}(t,0) \otimes \kappa(t,0)$. Hence

$$E_c(t) = D_x(W_{\mathrm{osc}} - V_1)^\natural(t,0) + D_x \bar{W}^\natural(t,0).$$

The field $W_{\mathrm{osc}} - V_1 = \sum_{p=2}^{P} V_p + \sum_{p=2}^{P+1} C_p$ is supported in $\operatorname{supp} g$. By (5.34) and (5.36), its norm $\|\cdot\|_{[1],\gamma}$ is at most $C_{2,P}\mathsf{D}^C\lambda^{-23/12}$, the bound for V_2, and $\|\bar{W}\|_{[1]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/2}$ by (5.37). The bound (5.44) at $m = 1$ therefore gives $|E_c| \le C_{2,P}\mathsf{D}^C\lambda^{-5/12} = C_{2,P}\mathsf{D}^C\lambda^{-1/12}\cdot\lambda^{-1/3}$. Since b is at least the exponent C of this bound and $\Gamma_P \ge C_{2,P}$, (5.43) makes it at most $\lambda^{-1/3}$. This proves (5.50). □

Estimating a flow by $\exp\int\|\nabla V\|_\infty$ would exponentiate the large shear. Even for the stationary shear in (6.2), the actual deformation grows only linearly in time. Write the new flow as $Y(t,a) = X(t,\mathcal{A}_t(a))$. The parent label $\mathcal{A}_t(a)$ of this particle moves, and so does its phase $k_*\cdot\mathcal{A}_t(a)$. Differentiating this composition at once would put the large normal k_* into the differential inequality for the flow. Instead we evolve the label and the phase as independent variables. Incompressibility then replaces the fast factor in the oscillatory phase velocity by a label derivative. The vector field has small gradient in $(a/\ell_*, s/h)$. Restriction to its phase graph reintroduces the factor $|k_*|/h$.

Lemma 5.10. *Let v be a smooth divergence-free field on $[0,T]\times\mathbb{T}^3$ with $\|v\|_{C_tC^2} \le \lambda^{-5/2}$. The flow Y of $V + v$ from time zero satisfies*

$$\sup|(D_aY)^{\pm1}| \le C_{\mathrm{step}}\lambda^{1/4}. \tag{5.53}$$

Proof. The field $v\circ X$ is independent of the phase, and $\mathbf{d}\cdot(v\circ X) = (\operatorname{div} v)\circ X = 0$, so $\widetilde{\mathbf{d}}\cdot(W + v\circ X) = 0$. By the chain rule and (5.8) for D_aX, $\|v\circ X\|_{[2]} \le C\mathsf{D}^C\|v\|_{C_tC^2}$. Put $c = (D_aX)^{-1}(W + v\circ X)$ and write $c = c_{\mathrm{osc}} + \bar{c}$ for its oscillatory and phase-independent parts, with c_{osc} repeated over the lattice by $(a,s)\mapsto(a+n, s+k_*\cdot n)$, $n\in\mathbb{Z}^3$. On the graph $s = k_*\cdot a$, the field c is then $(D_aX)^{-1}((W^\natural + v)\circ X)$. The Piola identity turns the solenoidality of $W + v\circ X$ into $(\nabla_a + k_*\partial_s)\cdot c = 0$, and the zero-mean phase primitive of its oscillatory part gives $k_*\cdot c_{\mathrm{osc}} = -\partial_s^{-1}\operatorname{div}_a c_{\mathrm{osc}}$. With (5.37) and (5.8), the norms without weight therefore satisfy

$$\|c_{\mathrm{osc}}\|_{[2]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/4}, \qquad \|\bar{c}\|_{[2]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/2}, \qquad \|k_*\cdot c\|_{[1]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/4};$$

in the last bound the oscillatory part costs one label derivative and the mean part the factor $|k_*| \le \mathsf{D}\lambda$ of (5.3). The flow estimate below uses only the order-one mean bound.

Let $\mathscr{L}_t$, with first component $a + \eta_a(t,a,s)$, be the flow from time zero of the label and phase velocities

$$\dot{a} = c(t,a,s), \qquad \dot{s} = k_*\cdot c(t,a,s). \tag{5.54}$$

The difference $s - k_*\cdot a$ is constant along this system, so the graph $s = k_*\cdot a$ is invariant. In the coordinates $(a/\ell_*, s/h)$ the vector field is $(c/\ell_*, k_*\cdot c/h)$, and its gradient is at most $C(\ell_*^{-1}\|c\|_{[1]} + h^{-1}\|k_*\cdot c\|_{[1]}) \le C_{2,P}\mathsf{D}^C\lambda^{-3/4} \le C_{2,P}\mathsf{D}^C\lambda^{-1/12}$. Since b is at least the exponent C of this bound, (5.43) bounds the gradient by $C_{2,P}/\Gamma_P$. Choose Γ_P large. Gronwall's inequality on $[0,T]$, $T \le 1$, bounds the derivatives of $\mathscr{L}_t$ and its inverse in the scaled coordinates by two. Integrating $\dot{\eta}_a = c$ along this flow gives

$$\ell_*|\partial_a\eta_a| + h|\partial_s\eta_a| \le C\|c\|_{[1]} \le C_{2,P}\mathsf{D}^C\lambda^{-5/4},$$

and the displacement of $\mathscr{L}_t^{-1}$ satisfies the same bound.

Differentiating $Y = X(t,\mathcal{A})$ gives $\partial_t\mathcal{A} = ((D_aX)^{-1}((W^\natural + v)\circ X))(t,\mathcal{A})$. Restricting (5.54) to its invariant graph gives $\mathcal{A}_t(a) = a + \eta_a(t,a,k_*\cdot a)$. The same formula with the displacement of

$\mathscr{L}_t^{-1}$ gives the inverse map. Thus $D\mathcal{A}_t - \mathrm{Id} = \partial_a\eta_a + \partial_s\eta_a \otimes k_*$, and with the corresponding identity for the inverse,

$$|D\mathcal{A}_t^{\pm 1} - \mathrm{Id}\,| \le \big(\ell_*^{-1} + |k_*|h^{-1}\big) C_{2,P}\mathsf{D}^C \lambda^{-5/4} \le C_{2,P}\mathsf{D}^C\lambda^{1/4}.$$

Then $D_aY = (D_aX \circ \mathcal{A})D\mathcal{A}$ and $(D_aY)^{-1} = (D\mathcal{A})^{-1}((D_aX)^{-1} \circ \mathcal{A})$, with the bound (5.1) for the deformation of the parent, give (5.53). □

5.7. **Frequency truncation.** The Fourier cutoff $S_\Lambda V$ gives the next parent finite frequency support. Bernstein's inequality then bounds each of its spatial derivatives. The tail $S_\Lambda V - V$ can extend across the whole torus, and cutting off $V \otimes V$ differs from multiplying the two cutoff velocities. The latter difference is the smoothing error also used in [6, Section 2.4, Proposition 2.2]. The profile pressure Π cancels terms in the expansion for V. For $u = S_\Lambda V$ we solve the pressure Poisson equation afresh. Recovering p from $S_\Lambda V$ gives a divergence-free residual. The equation for Π includes $\operatorname{div} R_V$, so both the old residual and the Fourier tail contribute to $D^2(p - \Pi)$. These estimates let Section 7 check $D^2p - \nabla R_u \le \mathrm{Id}$ for the next endpoint problem.

Proposition 5.11. *Let $\Lambda \ge 1$, and with the depth P of the expansion put*

$$u = S_\Lambda V, \qquad v_\Lambda = u - V, \qquad \varepsilon_\Lambda = \Lambda^{2-P}\big(1 + \|V\|_{C_tC^{P,1/2}}\big)^2. \tag{5.55}$$

Let p be the function of zero mean with $\Delta p = -\operatorname{tr}((\nabla u)^2)$, and let $R_u = u_t + u \cdot \nabla u + \nabla p$. Then u is smooth, odd, divergence free and supported on frequencies at most 2Λ, its residual R_u is odd, divergence free and supported on frequencies at most 4Λ, and

$$\|v_\Lambda\|_{C_tC^{2,1/2}} \le C_{2,P}\varepsilon_\Lambda, \qquad \|R_u\|_{C_tC^{1,1/2}} + \big\|D^2(p-\Pi)\big\|_{C_tC^{0,1/2}} \le C\,\|R_V\|_{C_tC^{1,1/2}} + C_{2,P}\varepsilon_\Lambda. \tag{5.56}$$

Moreover, on $[0,T] \times \mathbb{T}^3$,

$$D^2(p - p_U) \le \big(C_{\mathrm{step}}\lambda^{-1/4} + C\,\|R_V\|_{C_tC^{1,1/2}} + C_{2,P}\varepsilon_\Lambda\big)\,\mathrm{Id}\,. \tag{5.57}$$

Proof. By Lemma B.1, u and v_Λ are odd and divergence free and u is supported on frequencies at most 2Λ. Since $\operatorname{div}(u \cdot \nabla u) = \operatorname{tr}((\nabla u)^2)$, the residual is $R_u = \partial_t u + \mathbb{P}\operatorname{div}(u \otimes u)$; it is odd, divergence free and supported on frequencies at most 4Λ. Applying S_Λ and then $\mathbb{P}$ to (5.46) gives $\partial_t u + \mathbb{P}\operatorname{div} S_\Lambda(V \otimes V) = \mathbb{P}S_\Lambda R_V$. Subtracting this from the formula for R_u gives

$$R_u = \mathbb{P}\big[S_\Lambda R_V + \operatorname{div}\big(v_\Lambda \otimes u + V \otimes v_\Lambda + (\mathrm{Id} - S_\Lambda)(V \otimes V)\big)\big]. \tag{5.58}$$

To bound R_u in $C^{1,1/2}$ we need the three quadratic tensors in (5.58) in $C^{2,1/2}$. This accounts for the power Λ^{2-P} and the square in ε_Λ. The tail bound (B.2) at the order P gives $\|v_\Lambda\|_{C^{s,1/2}} \le C_{2,P}\Lambda^{s-P}\,\|V\|_{C^{P,1/2}}$ for $s \le 2$, and in particular the first bound of (5.56). By Leibniz' rule and the algebra inequality for the norm of $C^{0,1/2}$, for every integer $m \ge 0$,

$$\|fg\|_{C^{m,1/2}} \le C_m\,\|f\|_{C^{m,1/2}}\,\|g\|_{C^{m,1/2}}\,. \tag{5.59}$$

Hence the three tensors on the right of (5.58) are at most $C_{2,P}\Lambda^{2-P}\,\|V\|_{C^{P,1/2}}^2 \le C_{2,P}\varepsilon_\Lambda$ in $C^{2,1/2}$. The periodic multipliers $\mathbb{P}$ and $D^2\Delta^{-1}$ of order zero are bounded on $C^{0,1/2}$ and commute with derivatives, as does S_Λ; this bounds R_u in $C^{1,1/2}$. Subtracting the pressure equation for Π from the one for p gives

$$\Delta(p - \Pi) = -\operatorname{tr}(\nabla v_\Lambda \nabla V) - \operatorname{tr}(\nabla u \nabla v_\Lambda) - \operatorname{div} R_V,$$

and $\big\|D^2(p-\Pi)\big\|_{C^{0,1/2}} \le C\,\|v_\Lambda\|_{C^{1,1/2}}\,(\|V\|_{C^{1,1/2}} + \|u\|_{C^{1,1/2}}) + C\,\|R_V\|_{C^{1,1/2}}$. Taking the time suprema of the residual and pressure estimates separately and enlarging the constants proves (5.56). Adding $D^2(p - \Pi)$ to $D^2\mathsf{P}^\natural$ gives $D^2(p - p_U)$, so Lemma 5.9 gives (5.57). □

6. The initial wave and the inductive step

The first wave and all its successors must satisfy the same two inequalities (5.5). A stationary shear provides them for the first wave by direct calculation. The gradient of that wave then supplies a shear of its own. From the second wave onward, the comparison in Section 2 gives the two inequalities and a crossing at which the next wave can start.

6.1. The initial wave over a stationary shear.

For $0 < \rho \le 1$ let

$$U_0(x) = \sin(2\pi x_2)(e_1 + \rho e_3), \qquad B_\rho = 2\pi(e_1 + \rho e_3) \otimes e_2 \tag{6.1}$$

be the stationary shear of Section 1 and its gradient at the origin. The field U_0 is smooth, odd, divergence free and supported on the Fourier modes $\pm e_2$. As it depends only on x_2 and has no second component, $U_0 \cdot \nabla U_0 = 0$. Its pressure and residual therefore vanish. Each particle keeps its second coordinate, so the flow and its deformation are

$$X_0(t,a) = a + t\sin(2\pi a_2)(e_1 + \rho e_3), \qquad (D_a X_0)^{\pm 1}(t,a) = \mathrm{Id} \pm t\cos(2\pi a_2) B_\rho. \tag{6.2}$$

The inverse formula uses $B_\rho^2 = 0$. The origin is a fixed particle and $|B_\rho| \le 4\pi$.

For $\lambda \ge 1$ and $T > 0$, take the first wave over U_0 on $[0,T]$ with

$$k_* = \lambda e_1, \qquad \widetilde{\chi} \equiv 1, \tag{6.3}$$

and solve the seed equation (3.2) at the origin with initial direction e_2. Normalize its maximum on the whole interval $[0,T]$ to $\lambda^{-3/4}$. The launch amplitude is then determined by the solution of this ordinary differential equation. At time zero $\kappa(0,a) = \lambda e_1$ at every label, so the projection in (5.4) preserves that amplitude throughout the envelope. The initial frame at the origin is (e_1, e_2, e_3), in which

$$e_1 \cdot B_\rho e_2 = 2\pi, \qquad e_3 \cdot B_\rho e_2 = 2\pi\rho. \tag{6.4}$$

Thus ρ sets the first coupling ratio $\eta(0)/\nu(0)$. As $\widetilde{\chi} \equiv 1$, the rank-one term $\mathbf{a}(0,0) \otimes \kappa(0,0)$ in (5.50) is already present. The second wave can begin to switch on at the same instant.

Lemma 6.1. *If $T|B_\rho| \le 1/8$, the first wave satisfies the two conditions (5.5) for every $\mathsf{D} \ge 2$, and its strength at the origin at time zero lies between $\lambda^{1/4}/2$ and $\lambda^{1/4}$.*

Proof. At the origin the seed equation reads $\mathbf{b}_t = -(\mathrm{Id} - 2P_\kappa)B_\rho \mathbf{b}$. Since $\mathrm{Id} - 2P_\kappa$ is a reflection, every nonzero solution satisfies $\big|\partial_t \log|\mathbf{b}|\big| \le |B_\rho|$. Applying this estimate to $\mathbf{b}$ and to the seed $\mathbf{a}$, whose length is proportional to γ, gives

$$\frac{|\mathbf{b}(t)|}{\gamma(t)} \le e^{2T|B_\rho|}\frac{|\mathbf{b}(t')|}{\gamma(t')} \le 2\frac{|\mathbf{b}(t')|}{\gamma(t')} \qquad (0 \le t' \le t \le T).$$

The same estimate on the seed gives $e^{-T|B_\rho|}\lambda^{-3/4} \le |\mathbf{a}(0,0)| \le \lambda^{-3/4}$. Multiplying by $|\kappa(0,0)| = \lambda$ gives the initial strength.

For the sign of the coupling, the frame equations (2.2) give $|\partial_t E_1|, |\partial_t E_2| \le |B_\rho|$. Thus the first coupling can change by at most $2t|B_\rho|^2$ from its value in (6.4), and

$$\nu(t,0) = E_1(t) \cdot B_\rho E_2(t) \ge 2\pi - 2t|B_\rho|^2 \ge \pi.$$

□

6.2. **Choosing the next wave.** Let U and its flow X be as in Subsection 5.1. To launch a wave of frequency λ at t_0, we use the shear left by its predecessor. The origin is a fixed particle, so both waves sample the gradient there. Once the preceding cutoff equals one, (5.50) gives its rank-one contribution and (5.56) bounds the change due to truncation. Suppose, accordingly, that on $[t_0, T]$ the central gradient $M = \nabla U(t, 0)$ has the form

$$M = B + \mathbf{a}_p \otimes \kappa_p + E_c, \tag{6.5}$$

where B is the smooth trace-free gradient of the older velocity, $(\mathbf{a}_p, \kappa_p)$ is the preceding wave over B, with $\mathbf{a}_p \cdot \kappa_p = 0$ and $|\mathbf{a}_p||\kappa_p| > 0$ on $[t_0, T]$. The matrix E_c contains the errors from its corrections and truncation. The preceding pair solves (2.1) with $D = B$. In its moving frame (E_i) write $B_{ij} = E_i \cdot BE_j$. At t_0 freeze its strength $\sigma = |\mathbf{a}_p||\kappa_p|$ and its couplings $\nu = B_{12} > 0$ and $\eta = B_{32} > 0$. As in Subsection 2.2.1, put $r = \sqrt{2\sigma\nu}$, $S = \sigma/r$, $\varepsilon = \eta/\nu$ and $\tau = r(t - t_0)$.

Choose the wave vector at activation perpendicular to the preceding wave vector and amplitude. Since $\kappa = (D_a X)^{-T} k_*$, the time-zero vector that gives $\kappa(t_0, 0) = \lambda E_3(t_0)$ is

$$k_* = \lambda (D_a X(t_0, 0))^T E_3(t_0).$$

For the amplitude take the growing direction $-E_1(t_0)/S + E_2(t_0)$. In the coordinates of (2.3) this is $(\alpha_1, \alpha_2) = (-1, 1)$ at $\tau = 0$. Scale the seed by a positive constant so that $\max_{[t_0,T]} |\mathbf{a}(t, 0)| = \lambda^{-3/4}$, and write $\gamma(t) = \lambda^{3/4} |\mathbf{a}(t, 0)|$ as in Subsection 5.1.

To apply the comparison of Subsection 2.2, we need to bound the variation of the two couplings of B in the moving frame. When $\varepsilon \le 1$, the comparison error (2.6) satisfies

$$\delta \le C \left[\frac{(T - t_0)\big(\|B\|_\infty^2 + \|B_t\|_\infty\big)}{\eta} + \frac{\|B\|_\infty}{r} + \frac{\|E_c\|_\infty}{\nu} \right]. \tag{6.6}$$

Here the suprema are over $[t_0, T]$. Indeed, the frame equations (2.2) give $|\partial_t E_i| \le 2\|B\|_\infty$, so $|\partial_t B_{ij}| \le 4\|B\|_\infty^2 + \|B_t\|_\infty$. Integration from t_0 bounds the changes in B_{12} and B_{32}. Dividing by their frozen values ν and η, with $\eta \le \nu$, gives the first term of (6.6). The other two terms occur directly in (2.6).

The interval $[t_0, T]$ is measured in physical time. Its length in the variable τ of the comparison theorem is $r(T - t_0)$. Suppose this length is at least two and that (2.7) holds for that length, namely $C_0\varepsilon \le 1$ and $C_0[r(T - t_0)]^{C_0}\delta \le \varepsilon$. A positive rescaling of the seed preserves its frame, its couplings ν_+, η_+ over M and the ratio of its lengths at two times. Lemmas 2.6 and 2.7 therefore give, for $t_0 \le t' \le t \le T$ and every transverse solution $\mathbf{b}$ at the origin,

$$\frac{|\mathbf{b}(t)|}{\gamma(t)} \le CS \frac{|\mathbf{b}(t')|}{\gamma(t')}, \qquad \nu_+(t) \ge \tfrac{1}{2}\nu\varepsilon. \tag{6.7}$$

These are the two estimates (5.5) when D bounds CS and $2/(\nu\varepsilon)$.

We switch on the wave during the first unit of normalized time. With the function χ of Subsection 5.1, take

$$\widetilde{\chi}(t) = 1 - \chi\big(1 + r(t - t_0)\big) \quad (t \ge t_0), \qquad \widetilde{\chi}(t) = 0 \quad (t < t_0).$$

This cutoff is smooth, equals one for $\tau \ge 1$ and satisfies $|\widetilde{\chi}'| \le Cr$. Prescribe the coupling ratio $0 < \varepsilon_+ \le \varepsilon/32$ for the following wave, and suppose $r(T - t_0) \ge 2(\varepsilon\varepsilon_+)^{-1/2}$. By (2.24),

$$\gamma \le \exp\big(3 - \varepsilon^{-1/2}\big) \qquad \text{on } [t_0, t_0 + 1/r]. \tag{6.8}$$

Thus the factor $\|\widetilde{\chi}'\gamma\|_\infty$ in the residual estimate (5.48) is at most $Cr\exp(3 - \varepsilon^{-1/2})$.

Proposition 2.8 also gives a time in $\tau_m \le \tau \le 2(\varepsilon\varepsilon_+)^{-1/2}$ at which $\eta_+/\nu_+ = \varepsilon_+$. We activate the following wave at the first such crossing in this interval. Since the reference amplitude peaks at $\tau_m \ge 2/\varepsilon > 1$, the cutoff already equals one there, and (2.23) gives the strength of the shear available to amplify that wave.

7. A common sequence of approximate solutions

One initial frequency N will suffice for every step and every fixed derivative order. Start with the stationary shear (6.1), with $\rho = \varepsilon_1$. The schedule (1.14) keeps the order q_j fixed until the preceding frequency bounds the constants at the next order. It sets $\lambda_j = \lambda_{j-1}^{K_j}$ and, for $j \geq 2$, $L_j = 4 \log \lambda_{j+1}$. The resulting wave gain makes the switch-on error small at the frequency of wave $j+1$, whose first mean must cancel the parent residual. The target ratio $\varepsilon_j = L_{j+1}^{-2}$ sets the gain required of that next wave.

Write X_j for the flow of U_j from time zero, and π_j and R_j for its pressure and its residual, defined as p_U and R_U are for the parent in Subsection 5.1. At the center, let $\mathbf{a}_j, \kappa_j, \sigma_j$ be the seed, wave vector and strength of wave j, and let

$$\nu_j = \frac{\kappa_j}{|\kappa_j|} \cdot \nabla U_{j-1}(t,0) \frac{\mathbf{a}_j}{|\mathbf{a}_j|}, \qquad \eta_j = \frac{\kappa_j \times \mathbf{a}_j}{|\kappa_j||\mathbf{a}_j|} \cdot \nabla U_{j-1}(t,0) \frac{\mathbf{a}_j}{|\mathbf{a}_j|} \tag{7.1}$$

be the couplings of $\nabla U_{j-1}(t,0)$ in the frame of this pair. For $j \geq 2$ these are ν_+, η_+ of Subsection 6.2. Let $\widetilde{\chi}_j$ be the time cutoff of wave j and $O_j = W_{\mathrm{osc}}^{\natural}$ its oscillatory velocity increment before truncation, as in Lemma 5.8.

The first wave is present at time zero, when the second begins its activation. Given t_j, we allow a physical time τ_j for the next crossing and fix the end S_j of the mean equations by

$$t_1 = t_2 = 0, \qquad \tau_j = \lambda_{j-1}^{-1/8} L_{j+1}^3, \qquad S_0 = 1, \qquad S_j = t_j + 2\tau_j \quad (j \geq 1).$$

For $j \geq 2$ we first normalize the seed on this fixed $[t_j, S_j]$. The normalization leaves the couplings unchanged. We then define t_{j+1} by the first crossing of $\eta_j/\nu_j = \varepsilon_j$ in the bracket of Proposition 2.8. We shall show that the crossing occurs by $t_j + \tau_j$. The allowances decrease by a factor of at least four at each step (7.7), so the remaining launch intervals have total length at most $4\tau_j/3$. They all fit before S_j. The wave index j increases without bound. At each j we use only finitely many correction levels, from 1 to P. To make the residual small at the next frequency λ_{j+1}, take

$$P = 6K_{j+1}. \tag{7.2}$$

Since $\lambda_{j+1} = \lambda_j^{K_{j+1}}$, the depth error $\lambda_j^{-2P/3}$ is now λ_{j+1}^{-4} before multiplication by the physical derivative and parent factors.

In the theorem below, (T1) bounds the cost of differentiation through the Fourier support. The deformation and residual in (T2), together with the one-sided pressure estimate (T3), allow the endpoint inverse to be used again. At the origin (T4) supplies the shear and the crossing at which the next wave starts. The velocity allowance $\|U_j\|_\infty \leq \lambda_j$ serves the induction estimates. Boundedness of the limiting velocity comes from summing the increments in (7.4).

Theorem 7.1. *For every sufficiently large N there are fields U_j, $j \geq 1$, and times $0 = t_1 = t_2 < t_3 < \cdots$ with the following properties, in which U_0 is the shear (6.1) at $\rho = \varepsilon_1$:* (T1)–(T3) *for $j \geq 0$ and* (T4) *for $j \geq 1$.*

(T1) U_j is a smooth odd divergence-free field on $[0, S_j] \times \mathbb{T}^3$, supported on frequencies at most $2\Lambda_j$.

(T2) $\|U_j\|_{C_t L^\infty} \leq \lambda_j$, $\sup |(D_a X_j)^{\pm 1}| \leq \lambda_j$ and $\|R_j\|_{C_t L^\infty} \leq \lambda_{j+1}^{-3}$.

(T3) $D^2 \pi_j \leq \left(\sum_{i=1}^{j} \lambda_{i-1}^{-1}\right) \mathrm{Id}$ on $[0, S_j] \times \mathbb{T}^3$, as an upper bound of quadratic forms.

(T4) At the center on $[t_j, S_j]$, $\nabla U_j = \nabla U_{j-1} + \widetilde{\chi}_j \, \mathbf{a}_j \otimes \kappa_j + E_{c,j}$ with $|E_{c,j}| \leq \lambda_j^{-1/4}$. Moreover $0 \leq t_{j+1} - t_j \leq \tau_j$ and $\widetilde{\chi}_j = 1$ on $[t_{j+1}, S_j]$, so that the gradient of U_j has the splitting (6.5) there. At t_{j+1}, with an absolute constant $C \geq 2$ for which (2.23) holds,

$$\frac{\eta_j}{\nu_j} = \varepsilon_j, \qquad \pi \leq \nu_j \leq 4\pi, \qquad C^{-1} \lambda_j^{1/4} \leq \sigma_j \leq C \lambda_j^{1/4} L_{j+1}^2. \tag{7.3}$$

The activation times converge to a limit $T_\infty \in (0,\infty)$, and $T_\infty < S_j$ for every j. For $j \geq 1$ the oscillation O_j is supported, at each time of $[0,S_j]$, in the ball of radius λ_{j-1}^{-1} about the origin, and

$$\|O_j\|_{C([0,S_j];L^\infty)} \leq \lambda_j^{-1}, \qquad \|U_j - U_{j-1} - O_j\|_{C([0,S_j];C^{m,1/2})} \leq \lambda_j^{-1} \quad (0 \leq m \leq q_j). \tag{7.4}$$

For $j \geq 2$ the oscillation O_j vanishes on $[0,t_j]$.

For each fixed m, eventually $q_j \geq m$. From then on the remainders $U_j - U_{j-1} - O_j$ are summable in $C([0,T_\infty];C^{m,1/2})$ on the whole torus. The full increment has the same bound before activation or on a compact set outside the support of O_j. The oscillations themselves are summable globally in the supremum norm. Section 8 uses all three conclusions.

Figure 4 shows these intervals for three successive parents.

7.1. **The data of a step.** At any index j for which (T1) and (T2) hold, the frequency bound gives estimates for the gradient and its time derivative. As in Subsection 5.1, the residual is supported on frequencies at most $4\Lambda_j$ and

$$\partial_t U_j = -\mathbb{P}\,\mathrm{div}(U_j \otimes U_j) + R_j. \tag{7.5}$$

By (B.1), $\|\nabla U_j\|_\infty \leq C\Lambda_j\lambda_j$ and $\|\nabla R_j\|_\infty \leq C\Lambda_j\lambda_{j+1}^{-3}$. For the time derivative of the gradient, apply (B.1) to $U_j \otimes U_j$ and use the boundedness of $\mathbb{P}$ on $C^{0,1/2}$ in (7.5). Two spatial derivatives of the product in $C^{0,1/2}$ cost at most $C\Lambda_j^3\lambda_j^2 = C\lambda_j^{17}$. Thus for large N,

$$\|\nabla U_j\|_\infty \leq \lambda_j^9, \qquad \|\partial_t \nabla U_j\|_\infty \leq \lambda_j^{18} \quad \text{on } [0,S_j]. \tag{7.6}$$

At step j we apply Section 5 with $U = U_{j-1}$, $\lambda = \lambda_j$, activation time t_j and final time S_j. Write $\mu = \lambda_{j-1}$ and $\widetilde{\chi} = \widetilde{\chi}_j$. By (7.6) at $j-1$ we can take $\mathsf{D} = \mu^9$ for the parent gradient. When $j \geq 2$ we also use (7.6) at $j-2$ for the older gradient in the wave comparison. The final Fourier cutoff is $\Lambda_j = \lambda^5$.

To compare successive final times, observe that by (1.15), $\tau_j = \mu^{-1/8}L_{j+1}^3 \leq \mu^{-1/16}$ for large N. Applying this bound at $j+1$ and using $\lambda_j \geq \mu^4$ gives $4\tau_{j+1} \leq 4\mu^{-1/4} \leq \mu^{-1/8} \leq \tau_j$. Thus

$$\tau_j \leq \lambda_{j-1}^{-1/16}, \qquad 4\tau_{j+1} \leq \tau_j, \tag{7.7}$$

uniformly in j for large N.

The depth $P = 6K_{j+1}$ anticipates the next frequency. We need profile estimates through order $P+1$ now, even if q_j is much smaller. Since $P+1 \leq 7K(q_{j+1})$, $\mathcal{B}(q_{j+1})$ bounds all the fixed-order constants used in Section 5 and Lemma B.1 at these orders and this depth. The test defining q_{j+1} in (1.14) ensures that the preceding frequency μ already bounds $\mathcal{B}(q_{j+1})$. With $\mathsf{D} = \mu^9$ and $\lambda = \mu^{K_j}$, $K_j = 320b(q_j+1)$, we obtain

$$\mathcal{B}(q_{j+1})\mathsf{D}^b \leq \mu^{10b} \leq \lambda^{1/32}. \tag{7.8}$$

These inequalities verify the frequency window in Section 5. At step j, (7.8) gives $(\Gamma_P \mathsf{D}^b)^{12} \leq \lambda^{12/32} = \lambda^{3/8}$. On the other side, (T2) at step $j-1$ gives $\|R_{j-1}\|_{C_t L_x^\infty} \leq \lambda^{-3}$. For a nonzero residual the upper endpoint is therefore at least $\lambda^{6/5}$. Since $\lambda > 1$, the chosen frequency lies strictly between these two bounds.

7.2. **Proof of Theorem 7.1.** For $j = 0$, the shear U_0 has only the Fourier modes $\pm e_2$, $\|U_0\|_\infty \leq 2$, and its pressure and residual vanish. Its deformation and inverse have norm at most $1 + 4\pi$ on $[0,S_0]$ by (6.2). These give (T1)–(T3) for large N.

Suppose the construction has been carried out through step $j-1$. For $j = 1$, (7.7) gives $S_1 = 2\tau_1 \leq S_0 = 1$. For $j \geq 2$, the previous crossing occurred by $t_{j-1} + \tau_{j-1}$. Since $2\tau_j \leq \tau_{j-1}/2$,

$$S_j = t_j + 2\tau_j \leq t_{j-1} + \tfrac{3}{2}\tau_{j-1} < S_{j-1}.$$

Thus U_{j-1} is available on the entire interval $[0,S_j]$ of the endpoint problem.

(1) *Estimates for the parent.* On $[0, S_j]$, (T1) gives frequency support at most $2\mu^5$, and (7.6) gives $\|\nabla U_{j-1}\|_\infty \le \mu^9 = \mathsf{D}$. The deformation is at most μ by (T2). Its residual bound $\|R_{j-1}\|_\infty \le \lambda^{-3}$ implies (5.6). For coercivity we need one derivative of this residual. Bernstein's inequality at frequency $4\Lambda_{j-1}$ gives $\|\nabla R_{j-1}\|_\infty \le C\mu^5\lambda^{-3}$. The pressure sum in (T3) is at most $2/N$ since $\lambda_i \ge 2\lambda_{i-1}$. Thus $D^2\pi_{j-1} - \nabla R_{j-1} \le (2/N + C\mu^5\lambda^{-3})\,\mathrm{Id}$ as quadratic forms. Large N gives (5.2).

(2) *Choosing wave j.* For $j = 1$ take the wave (6.3) of frequency λ_1 on $[0, S_1]$. Since $4\pi S_1 \le 1/8$ for large N, Lemma 6.1 gives (5.5) and the strength bounds in (7.3) at $t_2 = t_1 = 0$. The couplings there are $\nu_1 = 2\pi$ and $\eta_1 = 2\pi\varepsilon_1$ by (6.4). The first wave has $\widetilde{\chi}_1 \equiv 1$.

For $j \ge 2$, (T4) at $j-1$ gives the splitting (6.5) on $[t_j, S_j]$ with $B = \nabla U_{j-2}(t, 0)$, the preceding pair $(\mathbf{a}_{j-1}, \kappa_{j-1})$ and $\|E_c\|_\infty \le \mu^{-1/4}$. The pair solves (2.1) over B because the flow of U_{j-2} fixes the origin. Freeze its couplings and strength at t_j. For $j = 2$ these are the values of wave 1 at time zero. For $j \ge 3$ they are the values at the preceding crossing. In both cases (7.3) gives

$$\varepsilon = \varepsilon_{j-1} = L_j^{-2}, \qquad \pi \le \nu \le 4\pi, \qquad c\mu^{1/4} \le \sigma \le C\mu^{1/4}L_j^2.$$

Choose the wave of Subsection 6.2, whose wave vector has length λ at t_j. Since $r = \sqrt{2\sigma\nu}$, $c\mu^{1/8} \le r \le C\mu^{1/8}L_j$. The normalized length $r(S_j - t_j) = 2r\mu^{-1/8}L_{j+1}^3$ is therefore between 2 and CL_{j+1}^4 for large N.

To check the comparison on this interval, (7.6) at $j-2$ gives $\|B\|_\infty \le \lambda_{j-2}^9 \le \mu^{1/32}$ and $\|B_t\|_\infty \le \mu^{1/16}$, since $\lambda_{j-2} = \mu^{1/K_{j-1}}$ and $K_{j-1} > 288$. With $\eta = \nu\varepsilon$, the three terms in (6.6) give

$$\delta \le C\big(\varepsilon^{-1}L_{j+1}^3\mu^{-1/16} + \mu^{-3/32} + \mu^{-1/4}\big).$$

By (1.15), $L_j \le L_{j+1} \le (\log\mu)^2$. Thus, with C_0 fixed as in Subsection 2.2, $C_0(2r\tau_j)^{C_0}\delta/\varepsilon$ is bounded by $\mu^{-1/16}$ times a fixed power of $\log\mu$, and $C_0\varepsilon \le C_0(\log N)^{-2}$. For this fixed C_0, both conditions of (2.7) hold for every $j \ge 2$ once N is sufficiently large; the threshold is independent of j. In the resulting central estimates (6.7), $CS = Cr/(2\nu) \le C\mu^{1/8}L_j$ and $2/(\nu\varepsilon) \le 2L_j^2/\pi$. Both are at most $\mu \le \mathsf{D}$ for large N. This proves (5.5) for wave j.

(3) *The next activation time.* We must place the next crossing in the first half of $[t_j, S_j]$. Indeed, $t_{j+1} \le t_j + \tau_j$ would give $S_{j+1} = t_{j+1} + 2\tau_{j+1} < S_j$ by (7.7). The parameter choice (1.14) gives $L_{j+1}/L_j = K_{j+2} \ge 640$, so

$$\varepsilon_j = (L_j/L_{j+1})^2\varepsilon_{j-1} \le \varepsilon_{j-1}/32, \tag{7.9}$$

and $(\varepsilon\varepsilon_j)^{-1/2} = L_jL_{j+1} \le L_{j+1}^2$. In physical time the upper endpoint of the bracket in Proposition 2.8 lies at most τ_j after t_j. Indeed,

$$\frac{2(\varepsilon\varepsilon_j)^{-1/2}}{r} \le C\mu^{-1/8}L_{j+1}^2 \le \tau_j,$$

using $r \ge c\mu^{1/8}$ and $L_{j+1} \ge \log N$. Take t_{j+1} to be the first crossing in that interval. Its normalized time is at least $\tau_m \ge 2/\varepsilon > 1$, so $0 < t_{j+1} - t_j \le \tau_j$ and $\widetilde{\chi} = 1$ on $[t_{j+1}, S_j]$.

At the crossing, the proof of Proposition 2.8 gives a wave vector of length comparable to λ/ε_j. The amplitude there is a fraction comparable to $(\varepsilon_j/\varepsilon_{j-1})^{1/2}$ of its maximum $\lambda^{-3/4}$ on $[t_j, S_j]$. Thus at $(t_{j+1}, 0)$,

$$|\kappa_j| \asymp \lambda L_{j+1}^2, \qquad |\mathbf{a}_j| \asymp \lambda^{-3/4}L_j/L_{j+1}.$$

Their product $\sigma_j(t_{j+1})$ is comparable to $\lambda^{1/4}L_jL_{j+1}$, as in (2.23). As $1 \le L_jL_{j+1} \le L_{j+1}^2$, this gives the strength bounds in (7.3). Equation (2.19) applies at the same time and gives

$$\big|\log\big(\nu_j(t_{j+1})/\nu_{j-1}(t_j)\big)\big| \le 2\varepsilon_{j-1}.$$

Summing this inequality over the successive crossings from $\nu_1(t_2) = 2\pi$ and using $\varepsilon_i \le 32^{1-i}\varepsilon_1$ from (7.9) gives $|\log(\nu_j(t_{j+1})/(2\pi))| \le 4\varepsilon_1 \le 4(\log N)^{-2}$. Thus $\pi \le \nu_j(t_{j+1}) \le 4\pi$ for large N, completing (7.3).

On the interval where the cutoff changes, (6.8) gives $\gamma \le e^{3-L_j} = e^3\lambda_{j+1}^{-4}$. Since $|\widetilde{\chi}'| \le Cr$ and $Ce^3 r \le \mu$ for large N by item 2,

$$\|\widetilde{\chi}'\gamma\|_\infty \le \mu\lambda_{j+1}^{-4}. \tag{7.10}$$

The same estimate holds for the first wave since $\widetilde{\chi}_1' \equiv 0$.

(4) *The expansion and its truncation.* Items 1 and 2 give the parent and wave assumptions of Subsection 5.1. By (7.8), $C_{\text{step}} = \Gamma_P \mathsf{D}^b \le \mu^{10b} \le \lambda^{1/32}$, so (5.43) holds. We can therefore apply the expansion and truncation estimates of Section 5.

Depth and gain now give the same power of the next frequency. For the expansion, $\lambda^{-2P/3} = \lambda_{j+1}^{-4}$ by (7.2). The switch-on bound (7.10) costs one extra factor μ. Since $C_{\text{step}}\mu \le \lambda^{1/16}$, the residual bound (5.48) gives

$$\|R_V\|_{C_tC^{1,1/2}} \le 2\lambda^{2+1/16}\lambda_{j+1}^{-4} \le \lambda_{j+1}^{-7/2}. \tag{7.11}$$

To estimate the Fourier tail in (5.55), we need $\|V\|_{C_tC^{P,1/2}}$. For the parent $U = U_{j-1}$, (T1), (T2) and the Bernstein bound (B.1) give $\|U\|_{C_tC^{P,1/2}} \le C_P\mu^{5(P+1)+1} \le \lambda^{P+2}$, since $C_P \le \mu$ and $\lambda = \mu^{K_j}$ with $K_j \ge 6$. For the wave, (7.8) bounds the factor $C_{P,P}\mathsf{D}^{b(P+2)}$ in (5.48) by $\lambda^{(P+2)/32}$. The larger of the two resulting exponents for the oscillation and the mean is $3P/2 + 1/4 + (P+2)/32$. This gives $\|W^\natural\|_{C_tC^{P,1/2}} \le \lambda^{2(P+1)}$ for large N. Hence $1 + \|V\|_{C_tC^{P,1/2}} \le 3\lambda^{2(P+1)}$. Inserting this with $\Lambda = \lambda^5$ and $9\mu \le \lambda$ gives

$$\mu\varepsilon_\Lambda \le 9\mu\lambda^{5(2-P)+4(P+1)} \le \lambda^{15-P} = \lambda^{15}\lambda_{j+1}^{-6} \le \lambda_{j+1}^{-5}. \tag{7.12}$$

In particular $C_{2,P}\varepsilon_\Lambda \le \lambda_{j+1}^{-5}$.

(5) *The new velocity.* Take $U_j = S_{\Lambda_j}V$ in Proposition 5.11. The proposition gives (T1), and (5.56) with (7.11) and (7.12) gives

$$\|R_j\|_{C_tL^\infty} \le C\lambda_{j+1}^{-7/2} + \lambda_{j+1}^{-5} \le \lambda_{j+1}^{-3}, \qquad \|v_\Lambda\|_{C_tC^2} \le \lambda_{j+1}^{-5}.$$

Since $U_j = V + v_\Lambda$ and $\lambda_{j+1}^{-5} \le \lambda^{-5/2}$, the divergence-free field v_Λ satisfies the hypothesis of Lemma 5.10. It follows that $\sup|(D_aX_j)^{\pm1}| \le C_{\text{step}}\lambda^{1/4} \le \lambda$.

By Proposition 5.4, W_{osc} is supported in $|a| \le 2\ell_*$ in label coordinates. The support of $O_j = W_{\text{osc}}^\natural$ lies in the image of this ball under the parent flow. The flow fixes the origin, so (5.1) and $\ell_* \le \mathsf{D}^{-3}$ place its Eulerian support in the ball of radius $2\mathsf{D}\ell_* \le 2\mathsf{D}^{-2} \le \mu^{-1}$. For $j \ge 2$, O_j vanishes before t_j by the same proposition. It vanishes also at t_j, since $\widetilde{\chi}(t_j) = 0$ and the higher oscillatory profiles have zero data there by Subsection 5.4. By (5.47) and (7.8), $\|O_j\|_{C_tL^\infty} \le \lambda^{1/32}\lambda^{-5/4} \le \lambda^{-1}$.

The rest of the velocity increment can extend across the torus. Since $V = U_{j-1} + W^\natural$,

$$U_j - U_{j-1} - O_j = \bar{W}^\natural + v_\Lambda.$$

For $0 \le m \le q_j$, the coefficient $C_{m,P}\mathsf{D}^{b(m+2)}$ in (5.48) is at most $\mu^{10b(m+2)} \le \lambda^{1/16}$, since $K_j = 320b(q_j+1)$. Thus $\|\bar{W}^\natural\|_{C_tC^{m,1/2}} \le \lambda^{1/16-5/2} \le \lambda^{-3/2}$. For the Fourier tail, (B.2) at order P and (7.12) give

$$\|v_\Lambda\|_{C_tC^{m,1/2}} \le \mu\Lambda_j^{m-P}\|V\|_{C_tC^{P,1/2}} \le \Lambda_j^{m-2}\mu\varepsilon_\Lambda \le \lambda^{5q_j}\lambda_{j+1}^{-5} \le \lambda_{j+1}^{-4},$$

where $K_{j+1} \ge K_j \ge 5q_j$. The sum of the mean and tail bounds is at most λ^{-1}, proving (7.4). In particular $\|U_j\|_{C_tL^\infty} \le \mu + 2\lambda^{-1} \le \lambda$ by (T2) for the parent. This completes (T2) for U_j.

(6) *Pressure and the central velocity gradient.* For the pressure increment, the leading term in (5.57) satisfies $C_{\mathrm{step}}\lambda^{-1/4} \le \lambda^{-7/32} \le \mu^{-2}$. By (7.11) and (7.12), the other two terms are at most $C\lambda_{j+1}^{-7/2} + \lambda_{j+1}^{-5}$. Hence $D^2\pi_j \le D^2\pi_{j-1} + \mu^{-1}\,\mathrm{Id}$ on $[0, S_j]$ for large N. Since $\mu^{-1} = \lambda_{j-1}^{-1}$ and $S_j \le S_{j-1}$, adding this increment to (T3) for the parent proves (T3) at step j.

At the origin truncation adds $\nabla v_\Lambda(t,0)$ to the gradient in (5.50). Thus the new error is $E_{c,j} = E_c + \nabla v_\Lambda(t,0)$ with $|E_{c,j}| \le \lambda^{-1/3} + \lambda_{j+1}^{-5} \le \lambda^{-1/4}$. From t_{j+1} onward the cutoff equals one, so the full rank-one term is present for the next wave. With the coupling and strength bounds from items 2 and 3, this proves (T4) and closes the induction.

The activation times increase strictly from $t_2 = 0$ onward. By (T4) their successive differences are bounded by τ_j, so (7.7) gives a finite limit $T_\infty \ge t_3 > 0$. For $j \ge 1$,

$$0 < T_\infty - t_j \le \sum_{p \ge j} \tau_p \le \tfrac{4}{3}\tau_j < S_j - t_j.$$

Thus $T_\infty < S_j$ for $j \ge 1$, and $T_\infty < S_0$ since $S_1 \le S_0$.

8. The limit and its terminal behavior

On the common interval $[0, T_\infty]$, the velocity increments have summable supremum norms, at most $2\lambda_j^{-1}$ by (7.4). Thus U_j converges uniformly on the closed space-time slab to a continuous field w. To pass to derivatives, fix first $T < T_\infty$. Sufficiently late oscillations have yet to start by T, so only the summable mean and Fourier corrections enter the tail on $[0, T]$. This gives smooth initial data. The residuals also tend to zero, so the integrated Euler equation passes to w. At T_∞ the other localization matters. Each compact set away from the origin misses all sufficiently late oscillations throughout $[0, T_\infty]$.

8.1. **Convergence before the accumulation time.**

Proposition 8.1. *The data $U_j(0)$ converge in every $C^{m,1/2}$ to an odd divergence-free field u_0. In particular, $u_0 \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$. Let u be the maximal smooth solution of the Euler equations from u_0, with maximal time $T_* > 0$, supplied by Section 1. Then $T_* \ge T_\infty$ and, for every $0 \le T < T_\infty$ and integer $m \ge 0$, $U_j \to u$ in $C([0,T];C^{m,1/2})$. The solution is smooth and odd on $[0,T_\infty)\times\mathbb{T}^3$.*

Proof. Fix $0 \le T < T_\infty$ and an integer $m \ge 1$. Since $t_j \uparrow T_\infty$ and $q_j \uparrow \infty$, we can choose j_0 with $T \le t_{j_0}$ and $q_{j_0} \ge m$. For $j > j_0$ the oscillation O_j vanishes on $[0,T]$. Thus (7.4) gives, for $\ell > n \ge j_0$,

$$\|U_\ell - U_n\|_{C([0,T];C^{m,1/2})} \le \sum_{j>n} \lambda_j^{-1} \le 2\lambda_{n+1}^{-1}.$$

The limit in each of these norms is w. In particular $u_0 = w(0)$ is smooth, and w is odd and divergence free on $[0,T]$.

For $0 \le t \le T$, integrating the projected equation (7.5) gives

$$U_j(t) = U_j(0) - \int_0^t \mathbb{P}\,\mathrm{div}(U_j \otimes U_j)\,\mathrm{d}t' + \int_0^t R_j\,\mathrm{d}t'.$$

The residual integral has supremum norm at most $T\lambda_{j+1}^{-3}$ by (T2). Convergence in $C([0,T];C^{1,1/2})$, the product bound (5.59) and boundedness of $\mathbb{P}$ on $C^{0,1/2}$ give uniform convergence of the nonlinear integrand. Passing to the limit and differentiating in t gives $\partial_t w = -\mathbb{P}\,\mathrm{div}(w \otimes w)$. Its right side is continuous in every spatial order. Successive time differentiations give joint smoothness. By uniqueness $w = u$ on their common interval, and the existence of w for every $T < T_\infty$ gives $T_* \ge T_\infty$. □

8.2. **Velocity, central vorticity and the maximal time.**

Proposition 8.2. *The solution has bounded velocity on* $[0, T_\infty)$*, and* $|\operatorname{curl} u(t,0)| \longrightarrow \infty$ *as* $t \uparrow T_\infty$.

Proof. The continuous field w on $[0, T_\infty] \times \mathbb{T}^3$ is bounded, and Proposition 8.1 identifies it with u before T_∞.

Large curl at the launches would still allow cancellation between them. Fix a large j and follow the origin throughout $[t_j, t_{j+1}]$. Wave $j-1$ is fully active while wave j switches on. We measure the curl in a direction in which the new wave cannot cancel its predecessor. Applying (T4) of Theorem 7.1 at steps $j-1$ and j gives, at the origin,

$$\nabla U_j = \nabla U_{j-2} + \mathbf{a}_{j-1} \otimes \kappa_{j-1} + \widetilde{\chi}_j \, \mathbf{a}_j \otimes \kappa_j + E_{c,j-1} + E_{c,j}.$$

A gradient $\mathbf{a} \otimes \kappa$ contributes $\kappa \times \mathbf{a}$ to vorticity. At each time we project $\operatorname{curl} U_j$ along the actual wave vector $\kappa_j(t,0)$, which removes the leading curl of wave j. The preceding wave contributes $\kappa_{j-1} \times \mathbf{a}_{j-1} = \sigma_{j-1} E_3$ in its own frame. By (2.3) and Lemma 2.4, the E_3 component of κ_j/λ_j is at least $1/2$ and its length is at most $C(1+\tau^2)$ at normalized time τ. The normalized length of $[t_j, S_j]$ is at most $C L_{j+1}^4$ by item 2 of the proof of Theorem 7.1. Thus

$$\frac{\kappa_j}{|\kappa_j|} \cdot (\kappa_{j-1} \times \mathbf{a}_{j-1}) \geq c \sigma_{j-1} L_{j+1}^{-8}, \qquad \frac{\kappa_j}{|\kappa_j|} \cdot (\kappa_j \times \mathbf{a}_j) = 0. \tag{8.1}$$

The size of the surviving component depends on σ_{j-1} throughout $[t_j, t_{j+1}]$. Put $\mu = \lambda_{j-1}$. At t_j, (7.3) gives $\sigma_{j-1} \geq c\mu^{1/4}$. In the frame of wave $j-1$, (2.2) identifies $\partial_t \log \sigma_{j-1}$ with the third diagonal entry of the older gradient ∇U_{j-2}. Thus $|\partial_t \log \sigma_{j-1}| \leq |\nabla U_{j-2}(t,0)| \leq \lambda_{j-2}^9 \leq \mu^{1/32}$, using (7.6) and item 2 above. Since the interval has length at most $\tau_j \leq \mu^{-1/16}$ by (T4) and (7.7), $\log \sigma_{j-1}$ changes by at most one. Hence $\sigma_{j-1} \geq c\mu^{1/4}$ on the whole interval. Since $L_{j+1}^8 \leq (\log \mu)^{16} \leq \mu^{1/8}$ for large j by (1.15), the projected curl of the preceding wave in (8.1) is at least $c\mu^{1/8}$. The older gradient contributes at most $C\lambda_{j-2}^9$ to the curl and the two central errors are at most $2\mu^{-1/4}$ in norm. Consequently

$$|\operatorname{curl} U_j(t,0)| \geq c\lambda_{j-1}^{1/8} - C\lambda_{j-2}^9 - C, \tag{8.2}$$

which tends to infinity uniformly on $[t_j, t_{j+1}]$.

For $\ell > j$ the oscillation O_ℓ vanishes throughout $[t_j, t_{j+1}]$. Applying (7.4) to the remaining increments at order one and taking the limit by Proposition 8.1 gives

$$\sup_{t_j \leq t \leq t_{j+1}} |\operatorname{curl}(u - U_j)(t,0)| \leq C \sum_{\ell > j} \lambda_\ell^{-1}.$$

This tail tends to zero. The successive intervals cover $[t_j, T_\infty)$, so (8.2) gives $|\operatorname{curl} u(t,0)| \to \infty$ as $t \uparrow T_\infty$. □

Corollary 8.3. *The maximal existence time is* $T_* = T_\infty \in (0, \infty)$*, and both integrals in* (1.2) *diverge.*

Proof. By Proposition 8.1, $T_* \geq T_\infty$. If $T_* > T_\infty$, then smoothness on the compact interval $[0, T_\infty]$ would bound $\operatorname{curl} u(t,0)$, contrary to Proposition 8.2. Hence $T_* = T_\infty$, which is positive and finite by Theorem 7.1. The initial data are smooth, so the periodic Beale–Kato–Majda criterion [9], in the formulation of [11, Theorem 1], implies $\int_0^{T_*} \|\operatorname{curl} u(t)\|_\infty \, \mathrm{d}t = \infty$. The gradient integral also diverges, since $\|\operatorname{curl} u\|_\infty \leq C \|\nabla u\|_\infty$. □

8.3. **The velocity away from the origin at the singular time.**

Proposition 8.4. *For every compact* $K \subset \mathbb{T}^3 \setminus \{0\}$ *and every integer* $m \geq 0$*, both* $u(t)$ *and* $\partial_t u(t)$ *converge in* $C^m(K)$ *as* $t \uparrow T_\infty$.

Proof. Fix K and m as in the statement. The support of O_j lies in the ball of radius λ_{j-1}^{-1} about the origin throughout $[0,T_\infty]$. For large j the ball is disjoint from K and $q_j \geq m$, so (7.4) gives $\|U_j - U_{j-1}\|_{C([0,T_\infty];C^m(K))} \leq \lambda_j^{-1}$. The increments are summable in this norm, hence $u(t) \to w(T_\infty)$ in $C^m(K)$.

For the time derivative, the projected equation samples the velocity even near the origin through the pressure. Choose a smooth cutoff χ equal to one near K and zero near the origin. On $[0,T_\infty)$,

$$\partial_t u = -\mathbb{P}\operatorname{div}(\chi u \otimes u) - \mathbb{P}\operatorname{div}\big((1-\chi)u \otimes u\big).$$

On a compact neighborhood of $\operatorname{supp}\chi$ the spatial convergence just proved holds through order $m+2$. Multiplication by χ therefore gives convergence of $\chi u \otimes u$ in $C^{m+1,1/2}$ on the torus. The bounded map $\mathbb{P}\operatorname{div} : C^{m+1,1/2} \to C^{m,1/2}$ gives convergence of the first term in $C^m(K)$.

For the second term, $\operatorname{supp}(1-\chi)$ is a positive distance from K. Only the pressure part $-\nabla\Delta^{-1}\operatorname{div}\operatorname{div}$ acts across this distance. Its kernel is smooth off the diagonal, so for any bounded tensor field A,

$$\|\mathbb{P}\operatorname{div}((1-\chi)A)\|_{C^m(K)} \leq C_{m,K,\chi}\|A\|_{L^1}.$$

Since $u(t)\otimes u(t) \to w(T_\infty)\otimes w(T_\infty)$ uniformly on the torus, the difference tends to zero in L^1. The estimate therefore gives convergence of the second term in $C^m(K)$. □

Propositions 8.1, 8.2 and 8.4 and Corollary 8.3 prove Theorem 1.1. For the smooth space-time extension away from the origin, the auxiliary note chooses $\mathcal{B}$ to control all fixed orders of time differentiation as well [31, Theorem 1.1].

8.4. **Further consequences for the central particle and the terminal flow.** *Time integral along the central particle.* The interval estimates give $\int_0^{T_*} |\operatorname{curl} u(t,0)|\,\mathrm{d}t = \infty$ by bounding both the vorticity and the duration of each interval. On a sufficiently late interval $[t_j, t_{j+1}]$, let r be the rate of wave j. The crossing occurs at normalized time $r(t_{j+1}-t_j) \geq \tau_m \geq 2/\varepsilon_{j-1} = 2L_j^2$. Item 2 of the proof of Theorem 7.1 gives $r \leq C\lambda_{j-1}^{1/8}L_j$. Thus the interval has length at least $cL_j\lambda_{j-1}^{-1/8}$. On this whole interval (8.2) and the tail estimate for $u - U_j$ give $|\operatorname{curl} u(t,0)| \geq c\lambda_{j-1}^{1/8}$. Multiplication yields

$$\int_{t_j}^{t_{j+1}} |\operatorname{curl} u(t,0)|\,\mathrm{d}t \geq cL_j \longrightarrow \infty.$$

Since $T_* - t_j \geq t_{j+1} - t_j$, the same bounds give $(T_*-t_j)|\operatorname{curl} u(t_j,0)| \geq cL_j \to \infty$ along the launch times. In particular $\limsup_{t\uparrow T_*}(T_*-t)|\operatorname{curl} u(t,0)| = \infty$. Along the launch times the vorticity exceeds every fixed multiple of $(T_*-t_j)^{-1}$. The unscaled vorticity estimate holds on the entire intervening intervals. For the original family with sufficiently large N, Jaswal obtains the stronger all-time rate $(T_*-t)^2|\omega(t,0)| \to \infty$ as $t \uparrow T_*$ [57, Theorem 1.5(c)].

Spatial regularity of the terminal velocity. The steep part of F_h has width $h = \lambda_j^{-1/2}$ in phase. Taking one spatial derivative of the evaluated profile costs $\lambda_j^{3/2}$ up to a power of the parent bound. Together with the velocity scale $\lambda_j^{-5/4}$ this determines the range of Hölder exponents. Indeed, for O_j on $[0,T_*]\times\mathbb{T}^3$, (5.37) and (5.44) at orders zero and one give

$$\|O_j\|_\infty \leq \lambda_{j-1}^A\lambda_j^{-5/4}, \qquad \|\nabla O_j\|_\infty \leq \lambda_{j-1}^A\lambda_j^{1/4}$$

for a fixed A and all large j. Here $\mathsf{D} = \lambda_{j-1}^9$, and $\mathcal{B}(q_{j+1}) \leq \lambda_{j-1}$ absorbs the constants at the chosen depth. Fix $0<\alpha<5/6$. Spatial interpolation and $\lambda_{j-1}^A = \lambda_j^{A/K_j}$ give

$$\sup_{0\leq t\leq T_*}\|O_j(t)\|_{C^\alpha(\mathbb{T}^3)} \leq C_\alpha\lambda_j^{-5/4+3\alpha/2+A/K_j}.$$

Since $K_j \to \infty$, the exponent is eventually bounded above by a negative constant depending on α. These norms are summable, and (7.4) sums the rest of the increments in $C([0,T_*];C^1(\mathbb{T}^3))$. The limit w thus extends u continuously to $[0,T_*]$ with values in $C^\alpha(\mathbb{T}^3)$, as also shown by Jaswal [57, Theorem 1.5(a)]. For the original construction with sufficiently large initial frequency parameter, Jaswal's Theorem 1.5(b) gives $w(T_*) \notin C^{5/6}$.

Energy of the terminal trace. The uniform convergence on the torus also carries the kinetic energy to the terminal trace $u_* = w(T_*)$. By energy conservation before T_*,

$$\int_{\mathbb{T}^3} |u_*|^2 \,\mathrm{d}x = \lim_{t \uparrow T_*} \int_{\mathbb{T}^3} |u(t)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^3} |u_0|^2 \,\mathrm{d}x.$$

Acceleration of the fluid particles. The one-sided pressure bound also controls the material acceleration everywhere before the singular time. By (T3) of Theorem 7.1 and the growth of the frequencies, $D^2\pi_j \le K\,\mathrm{Id}$ with $K = \sum_{i\ge 0} \lambda_i^{-1} \le 2/N$ for sufficiently large N. On each closed interval $[0,T]$ with $T < T_*$, the spatial convergence of Proposition 8.1 passes through the zero-mean pressure equation $-\Delta\pi_j = \partial_i\partial_k(U_j^i U_j^k)$, with repeated indices summed. Thus $D^2 p \le K\,\mathrm{Id}$ for the exact solution on this interval.

Fix x and a coordinate vector e_i. On the unit circle $g(s) = \partial_i p(t, x + se_i)$ has zero mean and $g' \le K$. For $0 \le s \le 1$, periodicity gives $g(0) - K(1-s) \le g(s) \le g(0) + Ks$. Integration gives $|g(0)| \le K/2$. Hence every material trajectory X of this solution satisfies

$$|X_{tt}(t,a)| = |\nabla p(t, X(t,a))| \le \sqrt{3}/N \qquad (0 \le t < T_*).$$

The estimate controls the acceleration along a particle. The spatial derivative of the velocity can still diverge there.

Appendix A. The Hölder inverse

Subsection 4.3 reduces the choice of initial velocity to the endpoint problem (4.9). We solve it on the normalized interval $I = [0,1]$ and estimate the velocity in Hölder norm. Write $\|f\|_k = \sup_{t\in I} \|f(t)\|_{C^{k,1/2}}$. Constants C, p depend only on the dimension.

Let $K(t,a)$ be symmetric and $\Omega(t,a)$ skew-symmetric, continuous in time with values in $C^{1,1/2}$. Set

$$(Jf)(t) = \int_0^t f(s)\,\mathrm{d}s.$$

Assume, for some $\kappa, \mu > 0$,

$$K \ge \kappa I, \tag{A.1}$$

$$\int_I \big(K(t,a)z_t \cdot z_t + \Omega(t,a)z_t \cdot z\big)\,\mathrm{d}t \ge \mu \int_I |z_t|^2\,\mathrm{d}t \quad (a \in \mathbb{T}^3,\ z \in H_0^1(I;\mathbb{R}^3)). \tag{A.2}$$

Put

$$Q = 2 + \kappa^{-1} + \mu^{-1} + \|K\|_1 + \|\Omega\|_1.$$

Theorem A.1. *Under (A.1)–(A.2), let $L \in C(I;C^{0,1/2})$. There are unique $z \in C^1(I;C^{0,1/2})$, $c \in C^{0,1/2}$ divergence free and $\nabla\phi \in C(I;C^{0,1/2})$ satisfying*

$$Kz_t - J(\Omega z_t) + L = c + \nabla\phi, \qquad \operatorname{div} z = 0, \qquad z(0) = z(1) = 0.$$

They obey

$$\|z_t\|_0 \le CQ^p \|L\|_0. \tag{A.3}$$

If K, Ω and L are smooth on $I \times \mathbb{T}^3$, then so are z, c and $\nabla\phi$.

Write $v = z_t$ and $z = Jv$. Since $z(1) = 0$, v has zero time mean at each label. The Leray projection $\mathbb{P}$ in labels removes $\nabla_a \phi$. Subtracting the time mean with $\mathcal{Q}$ removes c. With $\mathcal{R} = \mathbb{P}\mathcal{Q} = \mathcal{Q}\mathbb{P}$, the label velocity satisfies

$$\mathcal{R}Bv = -\mathcal{R}L, \qquad v = \mathcal{R}v, \qquad Bv = Kv - J(\Omega v).$$

For a path of zero time mean $w \in H = L^2(I;\mathbb{R}^3)$, we have $Jw \in H_0^1(I;\mathbb{R}^3)$ and $J^*w(t) = \int_t^1 w(s)\,\mathrm{d}s = -Jw(t)$. Hence at each label a,

$$\langle B(a)w, w\rangle_H = \int_I \big(Kw\cdot w + \Omega w \cdot Jw\big)\,\mathrm{d}t \geq \mu \|w\|_H^2$$

by (A.2) applied to Jw. Testing the projected equation against v and integrating over labels bounds v in $L^2_{a,t}$. To control its spatial increments, Lemma A.2 also uses the spatial variation of K and Ω. It gives $v \in C_a^{0,1/2}(L_t^2)$, still an estimate on whole time paths. A second inversion, of $v \mapsto \mathbb{P}(K(t)v)$ at each fixed time, supplies continuity in time and (A.3).

A.1. **Spatial freezing.** Both inversions have the form $\mathcal{R}(Bv) = f$ on the range of an orthogonal projection. Holding B fixed at one label a_0 decouples the spatial Fourier modes, even when $B(a_0)$ acts on a whole time path. Near a_0 the frozen inverse absorbs the error $B(a_0) - B(a)$. Farther away its kernel is smooth. Choosing the neighborhood radius as an inverse power of the coefficient and coercivity bounds gives a polynomial estimate for the variable-coefficient problem.

Lemma A.2. *Let either $H = \mathbb{R}^3$, $\mathcal{R} = \mathbb{P}$, or $H = L^2(I;\mathbb{R}^3)$, $\mathcal{R} = \mathbb{P}\mathcal{Q}$, where $\mathcal{Q}v = v - \int_0^1 v(t)\,\mathrm{d}t$. Let $B \in C^{1,1/2}(\mathbb{T}^3;\mathcal{L}(H))$ and $\mu > 0$. Suppose*

$$\langle B(a)w, w\rangle_H \geq \mu \|w\|_H^2 \tag{A.4}$$

for every $a \in \mathbb{T}^3$ and every $w \in H$, with $\mathcal{Q}w = w$ in the second case. Then $\mathcal{R}B$ is an isomorphism on $\mathcal{R}C^{0,1/2}(\mathbb{T}^3;H)$, and

$$\|v\|_{C_a^{0,1/2}(H)} \leq C(2 + \mu^{-1} + \|B\|_{C_a^1\mathcal{L}(H)})^p \,\|\mathcal{R}Bv\|_{C_a^{0,1/2}(H)}\,. \tag{A.5}$$

If moreover $B \in C^{k,1/2}(\mathbb{T}^3;\mathcal{L}(H))$ for an integer $k \geq 1$, the same operator is an isomorphism on $\mathcal{R}C^{k,1/2}(\mathbb{T}^3;H)$.

Proof. We first prove the a priori estimate and then obtain existence by continuation. For $\mathcal{R}v = v$, testing $f = \mathcal{R}(Bv)$ against v and using (A.4) gives

$$\mu \|v\|_{L^2_a(H)}^2 \leq \langle Bv, v\rangle_{L^2_a(H)} = \langle f, v\rangle_{L^2_a(H)}.$$

This bound will control the lower-order term in the Hölder estimate.

At a fixed label a_0, put $B_0 = B(a_0)$. Write $r(n)$ for the projection induced by $\mathcal{R}$ on the spatial Fourier mode n. Its Leray factor projects onto $n^\perp$ for $n \neq 0$ and fixes the zero mode. For time paths there is also the factor $\mathcal{Q}$. On the full space of Fourier coefficients the frozen operator extends to

$$A_0(n) = r(n)B_0 r(n) + I - r(n).$$

It acts as the identity on the orthogonal complement of $r(n)H$. Applying (A.4) to the real and imaginary parts of $r(n)w$ gives

$$\mathrm{Re}\langle A_0(n)w, w\rangle_H \geq \mu \|r(n)w\|_H^2 + \|(I - r(n))w\|_H^2 \geq \min(1,\mu)\,\|w\|_H^2\,.$$

Apply Lax–Milgram on the underlying real Hilbert space [14, §6.2.1, Theorem 1, pp. 315–317]. The bilinear form $(u,v) \mapsto \mathrm{Re}\langle A_0(n)u, v\rangle_H$ is coercive with constant $\min(1,\mu)$. This gives $\|A_0(n)^{-1}\| \leq \min(1,\mu)^{-1}$, even for nonsymmetric B_0.

The projection r is smooth on the unit sphere. The formula $D(A_0^{-1}) = -A_0^{-1}(DA_0)A_0^{-1}$ bounds its first five angular derivatives by a fixed polynomial in $2 + \mu^{-1} + \|B\|_\infty$. It follows that

$$\left\|A_0(D)^{-1} f\right\|_{C_a^{0,1/2}(H)} \le C_0 \|f\|_{C_a^{0,1/2}(H)}, \qquad C_0 \le C(2 + \mu^{-1} + \|B\|_\infty)^C. \tag{A.6}$$

Indeed, localize either symbol A_0^{-1} or $A_0^{-1} r$ to frequencies of size 2^j. Five integrations by parts on the rescaled annulus give the Euclidean kernel estimate

$$\|\nabla^d G_j(y)\|_{\mathcal{L}(H)} \le C C_0 \, 2^{(3+d)j} (1 + 2^j |y|)^{-5}, \qquad d = 0, 1.$$

For $d = 0$ the uniform L^1 bound persists under periodization. Applying the kernel to a slightly enlarged input block bounds the output block by $C C_0 \, 2^{-j/2} \|f\|_{C_a^{0,1/2}(H)}$. Summing differences with one derivative on the low blocks and the supremum bound on the high blocks gives (A.6). The zero mode acts on the mean with norm at most $\min(1, \mu)^{-1}$.

Only near a_0 is the coefficient difference $B_0 - B$ small. Choose a cutoff χ equal to one on the ball of radius r about a_0, supported in the ball of radius $2r$, with $0 \le \chi \le 1$ and $|\nabla \chi| \le C/r$. Splitting the source $(B_0 - B)v$ gives

$$\begin{aligned} v = {} & A_0(D)^{-1} f + A_0(D)^{-1} \mathcal{R}\{\chi (B_0 - B) v\} \\ & + A_0(D)^{-1} \mathcal{R}\{(1 - \chi)(B_0 - B) v\}. \end{aligned} \tag{A.7}$$

The first term is controlled by (A.6). The coefficient $\chi(B_0 - B)$ has supremum norm at most $Cr \|B\|_{C^1}$ and Lipschitz constant at most $C \|B\|_{C^1}$. Its 1/2-Hölder norm is therefore at most $Cr^{1/2} \|B\|_{C^1}$. Put $Q_0 = 2 + \mu^{-1} + \|B\|_{C^1}$. Choosing $r = Q_0^{-M}$ with M sufficiently large bounds the second term in $C_a^{0,1/2}(H)$ by $\frac{1}{4} \|v\|_{C_a^{0,1/2}(H)}$.

For the last term the source vanishes on the ball of radius r. The periodized bounds for $d = 1$ sum to $C C_0 (|y|^{-4} + 1)$ away from zero. At points within $r/2$ of a_0, integrating over torus distances $|y| \ge r/2$ bounds the gradient by $C C_0 r^{-1} \|B\|_\infty \|v\|_\infty$. For pairs of points at distance at most $r/2$, center the cutoff at one of them. The last term then contributes at most $C C_0 r^{-1/2} \|B\|_\infty \|v\|_\infty$ to the Hölder quotient. For other pairs use $C r^{-1/2} \|v\|_\infty$ for the quotient of v. Absorbing the second term of (A.7) gives

$$\|v\|_{C_a^{0,1/2}(H)} \le C_1 \big(\|f\|_{C_a^{0,1/2}(H)} + \|v\|_{L_a^\infty(H)} \big), \qquad 1 \le C_1 \le C Q_0^C.$$

For $0 < \varepsilon \le 1$, averaging on a ball of radius ε^2 gives

$$\|v\|_\infty \le \varepsilon \|v\|_{C_a^{0,1/2}(H)} + C \varepsilon^{-3} \|v\|_{L_a^2(H)}.$$

Taking $\varepsilon = (2C_1)^{-1}$ and using $\|v\|_{L_a^2(H)} \le \mu^{-1} \|f\|_{L_a^2(H)}$ from coercivity proves (A.5).

To obtain existence from this estimate, set $B_s = (1 - s) I + sB$. For $0 \le s \le 1$ its coercivity is at least $\min(1, \mu)$, so the estimate is uniform in s. When $B \in C^{k,1/2}$, apply it to $\partial^\nu v$ for $|\nu| \le k$. Spatial derivatives commute with $\mathcal{R}$, so

$$\mathcal{R} B_s \partial^\nu v = \partial^\nu (\mathcal{R} B_s v) - \mathcal{R}[\partial^\nu, B_s] v.$$

Only lower derivatives of v occur in the commutator. Induction therefore gives $\|v\|_{C^{k,1/2}} \le M_k \|\mathcal{R} B_s v\|_{C^{k,1/2}}$ uniformly in s. On $\mathcal{R} C^{k,1/2}$ the operator at $s = 0$ is the identity. The uniform estimate and the Neumann series extend its inverse in finitely many steps to $s = 1$, for $k = 0$ and for every admissible $k \ge 1$. □

A.2. **Recovering the velocity at each time.**

Proof of Theorem A.1. We first find a time path of zero mean at each label. On $H = L^2(I;\mathbb{R}^3)$ the coefficient $B = K - J\Omega$ satisfies $\|B\|_{C_a^1\mathcal{L}(H)} \le C(\|K\|_1 + \|\Omega\|_1)$. With the coercivity obtained from (A.2), Lemma A.2 gives a unique solution $v = \mathbb{P}\mathcal{Q}v$ of $\mathbb{P}\mathcal{Q}(Bv + L) = 0$, with

$$\|v\|_{C_a^{0,1/2}(L_t^2)} \le CQ^p \|L\|_0 . \tag{A.8}$$

Taking the time mean recovers the field removed by $\mathcal{Q}$,

$$c = \mathbb{P}\int_0^1 (Bv + L)\,\mathrm{d}t.$$

Then $\mathbb{P}(Bv + L) = c$ in $L_t^2 L_a^2$. In particular, for almost every time,

$$\mathbb{P}(Kv) = c + \mathbb{P}J(\Omega v) - \mathbb{P}L. \tag{A.9}$$

The right-hand side, denoted by $g(t)$, is already continuous in $C^{0,1/2}$. Indeed, (A.8) bounds c in $C^{0,1/2}$, and Cauchy–Schwarz on $[s,t]$ bounds the time increment of $J(\Omega v)$ in the same norm by $|t-s|^{1/2}\|\Omega v\|_{C_a^{0,1/2}(L_t^2)}$. Together with the hypothesis on L, this gives $\|g\|_0 \le CQ^p\|L\|_0$.

For every t let $S(t)u = \mathbb{P}(K(t)u)$ on $\mathbb{P}C^{0,1/2}$. By (A.1), Lemma A.2 with $H = \mathbb{R}^3$ bounds $S(t)^{-1}$ uniformly by CQ^p. Since K is continuous in time, the resolvent identity gives continuity of $S(t)^{-1}$ in operator norm. Thus $\widetilde{v}(t) = S(t)^{-1}g(t)$ is continuous in $C^{0,1/2}$ and satisfies (A.3). For almost every t, the difference $u = v(t) - \widetilde{v}(t)$ is solenoidal in L_a^2 and satisfies $\mathbb{P}(K(t)u) = 0$ by (A.9). Testing against u gives

$$\kappa\|u\|_{L_a^2}^2 \le \langle K(t)u, u\rangle_{L_a^2} = \langle \mathbb{P}(K(t)u), u\rangle_{L_a^2} = 0.$$

Thus v has the continuous representative $\widetilde{v}$ with the same zero time mean, so $z = Jv$ satisfies both endpoint conditions. At every time $\mathbb{P}(Bv + L - c) = 0$, hence $Bv + L - c$ is a periodic gradient $\nabla\phi \in C(I;C^{0,1/2})$. Uniqueness of the projected solution v gives uniqueness of z, c and $\nabla\phi$.

For smooth data the higher-order assertion of Lemma A.2 first gives $v \in C_a^{k,1/2}(L_t^2)$, hence $c \in C^{k,1/2}$ and $J(\Omega v) \in C(I;C^{k,1/2})$. The fixed-time inverse then gives $v \in C(I;C^{k,1/2})$ for each k. Since K is smooth, $S(t)^{-1}$ is smooth in operator norm on $\mathbb{P}C^{k,1/2}$. In (A.9), $J(\Omega v)$ gains one time derivative from v. Induction gives all time derivatives of v in every $C^{k,1/2}$ and thus smoothness of z and $\nabla\phi$. □

Appendix B. The frequency cutoff

Fix a radial $\psi \in C_c^\infty(\mathbb{R}^3)$ with $0 \le \psi \le 1$, $\psi = 1$ on $|\zeta| \le 1$ and $\operatorname{supp}\psi \subset \{|\zeta| \le 2\}$. For $\Lambda \ge 1$ let S_Λ multiply the spatial Fourier mode $e^{2\pi i\xi\cdot x}$, $\xi \in \mathbb{Z}^3$, by $\psi(\xi/\Lambda)$. For the next parent $u = S_\Lambda V$, differentiation preserves the frequency support $|\xi| \le 2\Lambda$. Multiplication adds frequencies, so the quadratic term has support $|\xi| \le 4\Lambda$. The Leray projection acts separately on each Fourier mode, and hence the same bound holds for the residual $\partial_t u + \mathbb{P}\operatorname{div}(u\otimes u)$. Estimate (B.1) now controls its derivatives. Estimate (B.2) controls the discarded velocity across the whole torus.

Lemma B.1. *With Λ fixed in time, S_Λ commutes with spatial and time derivatives of smooth fields, preserves parity and divergence freeness, and maps into fields supported on frequencies at most 2Λ. Its operator norms on L^∞ and on $C^{0,1/2}$ are bounded in terms of ψ alone. For every integer $m \ge 0$ and every normed-space-valued f supported on frequencies at most 4Λ,*

$$\sum_{|\beta|=m} \left\|\partial^\beta f\right\|_\infty \le C_m\Lambda^m\|f\|_\infty . \tag{B.1}$$

For all integers $0 \le s \le M$ and every $f \in C^{M,1/2}$,

$$\|(\mathrm{Id} - S_\Lambda)f\|_{C^{s,1/2}} \le C_M\Lambda^{s-M}\|f\|_{C^{M,1/2}} . \tag{B.2}$$

The constants C_m and C_M depend only on their index and on ψ.

Proof. The symbol is scalar, even and independent of time, so the cutoff preserves parity and divergence freeness and commutes with differentiation. Its support gives the frequency bound. For the norm estimates we use the convolution kernel as in the Euclidean Bernstein inequalities [55, Appendix A, (A.2)–(A.6)]. Let $K = \check{\psi}$, a Schwartz function, real-valued because ψ is real and even, and set $K_\Lambda(y) = \Lambda^3 K(\Lambda y)$. To compute $S_\Lambda f$, convolve the periodic extension of f to $\mathbb{R}^3$ with K_Λ. Since $\|K_\Lambda\|_{L^1} = \|K\|_{L^1}$, applying convolution to f and to its spatial increments gives the two operator bounds. If f has frequencies at most 4Λ, then $f = S_{4\Lambda} f$. The identity $\|\partial^\beta K_{4\Lambda}\|_{L^1} = (4\Lambda)^{|\beta|} \|\partial^\beta K\|_{L^1}$ gives (B.1) also for normed-space-valued f.

For (B.2) put $k = M - s$. When $k = 0$, apply the $C^{0,1/2}$ bound to the derivatives of f through order s. For $k \geq 1$, the kernel has integral one and all its positive-order moments vanish because ψ is constant near zero. Thus in $f - S_\Lambda f$ only the remainder after expanding $f(x-y)$ at $y = 0$ through degree $k-1$ contributes. Its $C^{s,1/2}$ norm in x is bounded by $C_M |y|^k \|f\|_{C^{M,1/2}}$, by Taylor's formula for each derivative through order s and its spatial increments. Integrating this bound against $|K_\Lambda(y)|$ gives (B.2), since $\int_{\mathbb{R}^3} |y|^k |K_\Lambda(y)| \, \mathrm{d}y = \Lambda^{-k} \int_{\mathbb{R}^3} |y|^k |K(y)| \, \mathrm{d}y$. □

Appendix C. The parameter inequalities

In the schedule of Section 1.5, the threshold $\mathcal{B}(s)$ bounds the constants fixed by the profiles, derivative order and expansion depth in Sections 3, 5, 6 and 7 and in Lemma B.1, for all orders and depths through $7K(s)$. There is no assumed growth law for these constants as s increases. The lower bound (1.13) is an additional condition on $\mathcal{B}$.

Proof of Lemma 1.3. The current order q_j is always admissible in the maximum (1.14). Indeed, $\mathcal{B}(q_1) = \mathcal{B}(1) \leq N = \lambda_0$, and for $j \geq 2$ the preceding choice gives $\mathcal{B}(q_j) \leq \lambda_{j-2} \leq \lambda_{j-1}$. Thus $q_{j+1} = q_j$ until λ_{j-1} reaches $\mathcal{B}(q_j + 1)$, at which point the order increases by one. Since $N > 1$ and $K_j \geq 2$, the frequencies tend to infinity. They eventually exceed every fixed threshold, so $q_j \to \infty$.

It remains to bound the logarithmic gain at the coming stages by the frequency already available at step j. Taking a logarithm turns three successive frequency powers into a product of three exponents. By (1.14),

$$L_{j+1} = 4 \log \lambda_{j+2} = 4 K_j K_{j+1} K_{j+2} \log \lambda_{j-1}.$$

The largest future order here is $q_{j+2} \leq q_{j+1} + 1$. The floor in (1.13) was chosen to bound this product even if both intervening steps raise the order. Indeed,

$$4 K_j K_{j+1} K_{j+2} \leq 4K(q_{j+1} + 1)^3 \leq \log \mathcal{B}(q_{j+1}) \leq \log \lambda_{j-1}.$$ □

Appendix D. Finite norms and coefficient estimates

Take $0 < \ell_* \leq 1$ and $h > 0$ as in Subsection 5.1. A label derivative can cost ℓ_*^{-1} and a phase derivative h^{-1}. Measuring derivatives in units of these lengths gives the product and differentiation rules below. When a label derivative hits a coefficient of a linear equation, it leaves fewer derivatives on the unknown. The factor ℓ_* on the differentiated coefficient then absorbs the cost of another application of the inverse. The final bound contains a single factor of the zero-order inverse norm and an order-dependent constant.

Write $|f|_m = \sum_{|\beta|=m} \sup |\partial^\beta f|$ for the spatial derivatives of order m, taken in the label a or the position x as appropriate. The supremum includes all variables. For a field $f(t, a, s)$ of period one in s and an integer $n \geq 0$, set

$$\|f\|_{[n]} = \sum_{m+j \leq n} \ell_*^m h^j |\partial_s^j f|_m, \qquad \|f\|_{[n],\gamma} = \|f/\gamma\|_{[n]}. \tag{D.1}$$

The weighted norm is taken on $[t_0, T]$, where $\gamma(t) > 0$ measures the amplitude of the seed at the origin. All norms may be restricted to a closed subinterval of their time domain.

For a phase-independent field $c(t,a)$, only the terms with $j=0$ occur in (D.1). The mean inverse of Theorem 4.2 uses spatial Hölder norms, so we put

$$|c|_{m,1/2} = \sum_{|\beta|=m} \sup_t \left\| \partial_a^\beta c(t) \right\|_{C^{0,1/2}}, \qquad \|c\|_{[n],1/2} = \sum_{m\le n} \ell_*^m |c|_{m,1/2}.$$

To pass from supremum bounds to this norm costs one derivative and a factor $\ell_*^{-1/2}$. Indeed, bounding $|v(x)-v(y)|$ by both $2\|v\|_\infty$ and $\|\nabla v\|_\infty |x-y|$ gives $[v]_{C^{0,1/2}} \le (2\|v\|_\infty \|\nabla v\|_\infty)^{1/2}$. Applying this to $\partial_a^\beta c$ and using $\ell_* \le 1$ gives

$$\text{(D.2)}\quad \ell_*^m |c|_{m,1/2} \le C\ell_*^{-1/2}\big(\ell_*^m |c|_m + \ell_*^{m+1}|c|_{m+1}\big), \qquad \text{hence} \quad \|c\|_{[n]} \le \|c\|_{[n],1/2} \le C\ell_*^{-1/2}\|c\|_{[n+1]}.$$

Lemma D.1. *For every integer $n \ge 0$ and fields f, g smooth in label and phase,*

$$\text{(D.3)}\qquad \sum_i \|\partial_{a_i} f\|_{[n],\gamma} \le C\ell_*^{-1}\|f\|_{[n+1],\gamma}, \qquad \|\partial_s f\|_{[n],\gamma} \le h^{-1}\|f\|_{[n+1],\gamma},$$

$$\text{(D.4)}\qquad \|fg\|_{[n],\gamma} \le C_n \|f\|_{[n]}\|g\|_{[n],\gamma}.$$

With $\langle f\rangle_s$ the phase average,

$$\text{(D.5)}\qquad \|\langle f\rangle_s\|_{[n],\gamma} \le \|f\|_{[n],\gamma}, \qquad \|f-\langle f\rangle_s\|_{[n],\gamma} \le 2\|f\|_{[n],\gamma}, \qquad \|\partial_s^{-1} f\|_{[n],\gamma} \le C\|f\|_{[n],\gamma} \quad (\langle f\rangle_s = 0).$$

For phase-independent fields, (D.4) also holds with $\|\cdot\|_{[n],1/2}$ in place of $\|\cdot\|_{[n]}$ and $\gamma \equiv 1$.

Proof. Each label derivative of order $m+1$ occurs at most three times in the sum over i in (D.3). The phase derivative shifts j by one in (D.1). For (D.4), apply Leibniz' rule to $f(g/\gamma)$, using that γ depends only on time. For the Hölder version use the product inequality in $C^{0,1/2}$.

Phase averaging has supremum norm at most one. For a zero-mean f, subtracting the average of $\int_0^s f(t,a,\sigma)\,\mathrm{d}\sigma$ gives its zero-mean periodic primitive, of supremum norm at most $2\sup|f|$. The phase operators commute with label and phase derivatives, so these bounds give (D.5) at every order. □

The coefficients in Subsections 5.3 and 5.4 are independent of the phase. Their label derivatives satisfy bounds of the form

$$\text{(D.6)}\qquad |c|_m \le C_m \mathsf{D}^{C(m+1)} \qquad (m \ge 0).$$

The absolute exponent C may depend on the coefficient. The next lemma allows us to multiply and invert such coefficients, and shows how much of the factor ℓ_*^m is available when at least one derivative falls on a coefficient.

Lemma D.2. *Let $\mathsf{D} \ge 1$ and let c, d be smooth phase-independent matrix fields that satisfy (D.6).*

(a) The product cd satisfies (D.6).

(b) If c is invertible and $\sup_{t,a}|c^{-1}|$ is at most a power of D with an absolute exponent, then c^{-1} satisfies (D.6).

(c) If $\ell_* \le \mathsf{D}^{-C}$ with the exponent C of (D.6) for c, then $\|c\|_{[n]} \le C_n \mathsf{D}^C$ for every $n \ge 0$, and $\ell_*^m |c|_m \le C_m \mathsf{D}^{2C}\ell_*$ for every $m \ge 1$.

Proof. For (a) and (b) take C large enough for the bounds on c, d and, in (b), on $\sup|c^{-1}|$. Leibniz' rule gives $|cd|_m \le C_m \mathsf{D}^{C(m+2)}$, proving (a) with exponent $2C$. For $e = c^{-1}$, differentiating $ce = I$ gives

$$|e|_m \le C_m \mathsf{D}^C \sum_{i=1}^m |c|_i |e|_{m-i} \qquad (m \ge 1).$$

Starting from $|e|_0 \le \mathsf{D}^C$, induction gives $|e|_m \le C_m \mathsf{D}^{C(3m+1)}$, proving (b) with exponent $3C$. For (c) use the exponent C for c in (D.6). Then $\ell_*^m |c|_m \le C_m \mathsf{D}^C (\ell_* \mathsf{D}^C)^m$. Since $\ell_* \mathsf{D}^C \le 1$, summing gives the first bound, and retaining one factor $\ell_* \mathsf{D}^C$ for $m \ge 1$ gives the second. □

For the mean we differentiate the integrated equation (4.9) in the unweighted Hölder norm. For the oscillations we differentiate (5.18) in the γ-weighted norm. Its unknown takes values in a fixed $\mathbb{R}^2$, so label differentiation preserves the space of unknowns. In both equations a derivative falling on a coefficient leaves fewer derivatives on the solution. The following estimate uses this fact to retain just one factor of the zero-order inverse bound.

Lemma D.3. *Consider a linear equation of the form*

$$\mathcal{L}v = \mathcal{L}_0 v + \sum_\nu T_\nu(c_\nu v) = F$$

with finitely many phase-independent coefficients $c_\nu(t,a)$. Assume that $\mathcal{L}_0$ and the linear operators T_ν commute with label and phase derivatives. The equation may hold modulo a fixed subspace, such as gradients. Assume that differentiation preserves this subspace and the linear constraints on v, and commutes with the maps giving any prescribed data. Write f for the forcing F together with these data, with the sum of their norms. Suppose that every admissible solution, with its own forcing and prescribed data, satisfies the estimate

$$\|v\|_{[0],\gamma} \le C_{\mathrm{low}} \|f\|_{[0],\gamma}.$$

Let b_ν be the operator norm of T_ν from $\|\cdot\|_{[0],\gamma}$ to the forcing norm. If

$$C_{\mathrm{low}} \sum_\nu b_\nu \ell_*^m |c_\nu|_m \le C_m \qquad (m \ge 1), \tag{D.7}$$

then every smooth solution satisfies, for $n \ge 0$,

$$\|v\|_{[n],\gamma} \le C_n C_{\mathrm{low}} \|f\|_{[n],\gamma}.$$

Here C_n depends only on n and the bounds C_m in (D.7) for $m \le n$. For phase-independent fields with $\gamma \equiv 1$, the same assertion holds using $\|\cdot\|_{[n],1/2}$ and $|c_\nu|_{m,1/2}$ throughout, with b_ν taken in the corresponding zero-order norms.

Proof. Phase derivatives commute with the equation. For a label multi-index β and phase order j, the differentiated forcing is

$$\mathcal{L}(\partial_a^\beta \partial_s^j v) = \partial_a^\beta \partial_s^j F - \sum_\nu \sum_{0<\beta' \le \beta} \binom{\beta}{\beta'} T_\nu\big((\partial_a^{\beta'} c_\nu)\partial_a^{\beta-\beta'}\partial_s^j v\big).$$

The zero-order estimate applies with the differentiated data. Multiply by $\ell_*^{|\beta|} h^j$ and sum over $|\beta| + j \le n$. A term with $i = |\beta'| \ge 1$ assigns ℓ_*^i to the coefficient and leaves a norm of order at most $n - i$ on v. Thus for $n \ge 1$,

$$\|v\|_{[n],\gamma} \le C_{\mathrm{low}} \|f\|_{[n],\gamma} + C_n \sum_{i=1}^{n} \Big(C_{\mathrm{low}} \sum_\nu b_\nu \ell_*^i |c_\nu|_i \Big) \|v\|_{[n-i],\gamma}.$$

By (D.7), the sum is bounded by $C_n \|v\|_{[n-1],\gamma}$. Induction from order zero gives the claim. For the Hölder norms the same argument uses the product inequality in $C^{0,1/2}$. □

Apply Lemma D.3 with the zero-order inverse bounds from Theorem 4.2 and (5.23). When at least one derivative falls on a coefficient, Lemma D.2(c) leaves one factor ℓ_* in $\ell_*^m |c_\nu|_m$. In the Hölder estimate half a power remains by (D.2), as in (5.13). Either gain can absorb $C_{\mathrm{low}} \sum_\nu b_\nu$, which grows at most as a fixed power of D. Fix the exponents in (D.6) first. Then one sufficiently large A in $\ell_* = \mathsf{D}^{-A}$ gives (D.7) at every order, with constants C_m.

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The authors' account of the construction

The authors describe the origins of their argument as follows.

Before OpenAI released its paper on the unforced Euler equations, the first and second authors had completed a second proof of blow-up for the Euler equations with smooth forcing. Unlike their public proof [2], which uses axisymmetric flow with swirl, this unreleased proof uses the same Craik–Criminale amplification mechanism as OpenAI's paper, including the same reduced equations for the amplitude of a wave, the system (2.5) above. In our unpublished work on the hypodissipative Navier–Stokes equations, much of the argument for removing the force was developed, with human prompts guiding that work. This is consistent with the second author's statement of September 7 that the hypodissipative work pointed toward a blow-up construction for the unforced Euler equations [30].

The second author's account of his communications with OpenAI about these results is given in [30].

The authors' statement on AI

The following is the authors' account of how they wrote their source manuscript.

We used Claude and Codex throughout this work. The first proof without force was written by Anthropic models from the drafts on the Euler equations with forcing and on the hypodissipative Navier–Stokes equations described above. Like the first writeups of our proofs for the Boussinesq and Euler equations, it was long and hard to read. We then rewrote it with Claude and Codex under our direction, iterating with the models on the structure of the argument, isolating the key ideas, and simplifying it over successive revisions. In the course of this we replaced parts of the argument by ideas of our own, which the models implemented. Together with our own editing, this led to the present text.

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